

Elliptic Lie superalgebras of rank ≥ 2

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Abstract

In this paper, we introduce a class of Lie superalgebras \mathfrak{g} , defined with generators and relations. We show that non-isotropic roots of our Lie superalgebra \mathfrak{g} form a non-reduced elliptic root system R of rank ≥ 2 in the sense of K. Saito [13]. We also show that \mathfrak{g} is universal among Lie superalgebras associated with R .

Keywords: elliptic Lie algebras; toroidal Lie algebras; extended affine Lie algebras; elliptic Lie superalgebras

Contents

Introduction

1. Main results (Theorem 1.1 (1)-(3)) and strategy
2. Facts of the affine root systems
3. Elliptic root systems of rank ≥ 2
4. Root space decomposition (Proof of Theorem 1.1 (1))
5. Examples of representations (Proof of Lemma 4.1)
6. Invariant form (Proof of Theorem 1.1 (2))
7. Universality (Proof of Theorem 1.1 (3))
8. Proofs of well-known facts
9. Dynkin diagrams of the affine root systems

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References

Introduction

In 1985, K. Saito [13] introduced the notion of *n-extended affine root systems*. If $n = 0$ (respectively, $n = 1$), it is an irreducible finite root system

(respectively, an affine root system). In [13], he also intensively studied 2-extended affine root systems, which are now called *elliptic root systems* (see [14]). Since then, various attempts have been made to construct Lie algebras whose non-isotropic roots form those root systems. Among them are *toroidal Lie algebras* [12] and *extended affine Lie algebras* [1]. See [15, Introduction] for the history.

In 2000, K. Saito and D. Yoshii [15] constructed Lie algebras by using the Borcherds lattice vertex algebras, called them *simply-laced elliptic Lie algebras* and showed that they are isomorphic to the *ADE*-type (2-variable) toroidal Lie algebras of rank ≥ 2 . They also gave two other definitions for their Lie algebras. One uses generators and relations. The other uses (affine-type) Heisenberg Lie algebras; this was generalized by D. Yoshii [19] in order to define Lie algebras associated with the reduced elliptic root systems, and he called them *elliptic Lie algebras*. In 2004, the author [18] gave defining relations of the elliptic Lie algebras of rank ≥ 2 (see also Remark 1.2).

In this paper, we introduce a class of Lie superalgebras \mathfrak{g} defined with generators and relations (see Definition 1.3). We also show that non-isotropic roots of \mathfrak{g} form a non-reduced elliptic root system R of rank ≥ 2 (see Theorem 1.1 (1)) and there exists a natural super-symmetric invariant form on \mathfrak{g} (see Theorem 1.1 (2)). Moreover, we show *universality* of \mathfrak{g} , which means that there exists a natural epimorphism $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ if the non-isotropic roots of a Lie superalgebra \mathfrak{g}' form R (see Theorem 1.1 (3)).

We point out that our defining relations are closely related to defining relations, called *Drinfeld realization*, of the quantum affine algebras due to V.G. Drinfeld [5, Theorems 3 and 4].

The paper is organized as follows. In Section 1, we state our main theorem, Theorem 1.1 (1)-(3), and explain strategy of the proof of it in Subsection 1.8. In Section 2, we mention some basic facts on the affine root systems. In Section 3, especially in Theorem 3.1, we use maps k and g to give an expression (3.3) of the elliptic root systems of rank ≥ 2 (see also (1.18)); the expression is a key to this paper. In Section 4, at first, by Lemma 4.1, we evaluate the dimension of the Cartan subalgebra \mathfrak{h} of \mathfrak{g} , and finally we prove Theorem 1.1 (1). In Section 5, we prove Lemma 4.1; a strategy of the proof of it is given in Subsection 5.1. In Section 6, we prove Theorem 1.1 (2). In Section 7, we prove Theorem 1.1 (3). In Section 8, we give (self-contained) proofs of some of the basic and well-known facts used in the previous sections; especially we prove Theorem 2.1, which explains the existence of a base of an affine root system using the one of an irreducible finite root system (cf.

MacDonald [11]). In Section 9, we give a list of the Dynkin diagrams of the (reduced and non-reduced) affine root systems, which are helpful in reading this paper.

1 Main results (Theorem 1.1 (1)-(3)) and strategy

1.1 Basic Notation

Let $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x \geq 0\}$ and $\mathbb{Z}_- = \{x \in \mathbb{Z} | x \leq 0\}$. Then $\mathbb{N} \cup \{0\} = \mathbb{Z}_+$.

Let \mathbb{X} be a subring (with 1) of \mathbb{C} and \mathbb{M} a \mathbb{X} -module. Let \mathbb{Y} be a subset of \mathbb{X} . For $m \in \mathbb{M}$, set $\mathbb{Y}m := \{ym | y \in \mathbb{Y}\}$. For a subset \mathbb{S} of \mathbb{M} with $|\mathbb{S}| \geq 2$, set $\mathbb{Y}\mathbb{S} := \sum_{s \in \mathbb{S}} \mathbb{Y}s$. ($|\mathbb{S}|$ is the cardinal number of \mathbb{S} .)

Throughout of this paper, for any \mathbb{F} -linear space \mathbb{V} with a symmetric bilinear form $(,) : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{F}$, where \mathbb{F} is \mathbb{R} or \mathbb{C} , we set $\mathbb{V}^0 := \{v \in \mathbb{V} | (v, v) = 0\}$ and $\mathbb{V}^\times := \mathbb{V} \setminus \mathbb{V}^0$; for each $v \in \mathbb{V}^\times$, we set $v^\vee := \frac{2v}{(v, v)}$ and define $s_v \in \text{GL}(\mathbb{V})$ by $s_v(w) = w - (v^\vee, w)v$ ($w \in \mathbb{V}$); for any non-empty subset S of \mathbb{V}^\times , we denote by W_S the subgroup of $\text{GL}(\mathbb{V})$ generated by $\{s_v | v \in S\}$, i.e.,

$$(1.1) \quad W_S := \langle s_v | v \in S \rangle.$$

1.2 Extended affine root systems

Definition 1.1. Let $l \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Let V be an $(l + n)$ -dimensional \mathbb{R} -linear space. Recall V^0 and V^\times from Subsection 1.1. Assume that there exists a positive semi-definite symmetric bilinear form $(,) : V \times V \rightarrow \mathbb{R}$ such that $\dim V^0 = n$. Let R be a subset of V . Then R (or more precisely, (R, V)) is an (n) -extended affine root system if R satisfies the following axioms:

- (AX1) $R \subset V^\times$, $V = \mathbb{R}R$.
- (AX2) $\text{rank}_{\mathbb{Z}} \mathbb{Z}R = n + l (= \dim_{\mathbb{R}} V)$.
- (AX3) $(\alpha^\vee, \beta) \in \mathbb{Z}$ for $\alpha, \beta \in R$.
- (AX4) $s_\alpha(R) = R$ for all $\alpha \in R$.
- (AX5) If $R = R_1 \cup R_2$ and $(\alpha, \beta) = 0$ for $\alpha \in R_1$ and $\beta \in R_2$, then $R_1 = \emptyset$ or $R_2 = \emptyset$.

(see [13, (1.2) Definition 1 and (1.3) Note 2 iii].) For $\mu \in R$, let

$$(1.2) \quad p(\mu) = \begin{cases} 0 & \text{if } 2\mu \notin R, \\ 1 & \text{if } 2\mu \in R \end{cases}$$

(see (1.3) below). Let W be the subgroup of $\text{GL}(V)$ generated by $\{s_\alpha | \alpha \in R\}$, i.e., $W = W_R$ (see (1.1)).

Let R be an extended affine root system. It is well-known that for all $\alpha \in R$,

$$(1.3) \quad \begin{cases} R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}, \{\alpha, 2\alpha, -\alpha, -2\alpha\} \text{ or } \{\alpha, \frac{1}{2}\alpha, -\alpha, -\frac{1}{2}\alpha\}, \\ \text{(so } -R = R). \end{cases}$$

We call R *reduced* (resp. *non-reduced*) if $R \cap 2R = \emptyset$ (resp. $R \cap 2R \neq \emptyset$).

We say that two extended affine root systems (R, V) and (R', V') are *isomorphic* if there exist an \mathbb{R} -linear bijective map $f : V \rightarrow V'$ and $c \in \mathbb{R}$ with $c > 0$ such that $f(R) = R'$ and $(f(v), f(w)) = c(v, w)$ for $v, w \in V$.

By (AX2), there exists a basis $\{x_1, \dots, x_l, a_1, \dots, a_n\}$ of $\mathbb{Z}R$ such that $\{a_1, \dots, a_n\}$ is a basis of $V^0 \cap \mathbb{Z}R$; so, for $\mathbb{X} = \mathbb{Z}$ or \mathbb{R} , we have

$$(1.4) \quad \mathbb{X}R = \left(\bigoplus_{i=1}^l \mathbb{X}x_i \right) \bigoplus \left(\bigoplus_{j=1}^n \mathbb{X}a_j \right) \text{ and } V^0 \cap \mathbb{X}R = \bigoplus_{j=1}^n \mathbb{X}a_j.$$

In this paper, we always let

$$(1.5) \quad \pi : V \rightarrow V/V^0$$

denote the canonical map.

Let R , l and n be as in Definition 1.1. We call l (resp. n) the *rank* (resp. *nullity*) of R . If $n = 0$, then R is an *irreducible finite root system* (see [13, (1.3) Example 1 i]). If $n = 1$, then R is an *affine root system* (see [13, (1.3) Example 1 ii]). If $n = 2$, then R is an *elliptic root system* (see [13, (1.3) Example 1 iii]), [14] and [15]).

Let R be an irreducible finite root system or an affine root system (i.e., $n \in \{0, 1\}$). Then a subset Π of R formed by $(l + n)$ -linearly independent elements is called a *base* if

$$(1.6) \quad R = (R \cap \mathbb{Z}_+\Pi) \cup (R \cap \mathbb{Z}_-\Pi).$$

(For $n = 0$, see [6, Theorem 10.1]. For $n = 1$, see Theorem 2.1 (1) (1-1) in this paper (cf. MacDonald [11, (4.6)] (see also [13, (3.3) i)-iii])). If Π is a base of R , then, for $\mathbb{X} = \mathbb{Z}$ or \mathbb{X} , we have

$$(1.7) \quad \mathbb{X}R = \bigoplus_{\alpha \in \Pi} \mathbb{X}\alpha.$$

Let R be an affine root system (i.e., $n = 1$) with a base $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_l\}$; in this paper, we always assume α_0 to be such that $\{\pi(\alpha_1), \dots, \pi(\alpha_l)\}$ is a base of $\pi(R)$ (see Theorem 2.1 (4)). Let $\delta(\Pi) \in \mathbb{Z}\Pi$ be such that

$$(1.8) \quad \delta(\Pi) \in \mathbb{N}\Pi \text{ and } \mathbb{Z}\delta(\Pi) = \mathbb{Z}R \cap V^0$$

(see Theorem 2.1 (4)). Let $\delta = \delta(\Pi)$. By (1.4) and (1.7), for $\mathbb{X} = \mathbb{Z}$ or \mathbb{R} , we have

$$(1.9) \quad \mathbb{X}R = \bigoplus_{\alpha \in \Pi} \mathbb{X}\alpha = \left(\bigoplus_{\alpha \in \Pi \setminus \{\alpha_0\}} \mathbb{X}\alpha \right) \bigoplus \mathbb{X}\delta \text{ and } \mathbb{X}\delta = \mathbb{X}R \cap V^0.$$

Set

$$(1.10) \quad A(\Pi) := ((\alpha_i^\vee, \alpha_j))_{0 \leq i, j \leq l}.$$

Then $A(\Pi)$ is a generalized Cartan matrix of affine-type (see Theorem 2.1 (4)). If $\tilde{\Pi}$ is another base of R , then $\delta(\tilde{\Pi}) = \delta(\Pi)$ or $-\delta(\Pi)$ and $A(\tilde{\Pi}) = A(\Pi)$ (after renumbering $\alpha_1, \dots, \alpha_l$) (see Theorem 2.1 (3)-(4)).

1.3 Fundamental-set

Definition 1.2. (*Fundamental-set* $\Pi \cup \{a\}$) Let (R, V) be an elliptic root system (i.e., $n = 2$) of rank $l \geq 2$. A subset $\Pi \cup \{a\}$ of $\mathbb{Z}R$ is called a *fundamental-set* of R if it satisfies the axioms (FS1)-(FS3) below; in this paper, we always let

$$(1.11) \quad \pi_a : V \rightarrow V/\mathbb{R}a$$

denote the canonical map.

(FS1) $a \in (\mathbb{Z}R \cap V^0) \setminus \{0\}$ and there exists $b \in \mathbb{Z}R \cap V^0$ such that $\{a, b\}$ is a basis of $\mathbb{Z}R \cap V^0$, i.e., $\mathbb{Z}R \cap V^0 = \mathbb{Z}a \oplus \mathbb{Z}b$.

(FS2) $|\Pi| = l + 1$, $\Pi \subset R$ and $\pi_a(\Pi)$ is a base of the affine root system $\pi_a(R)$.

(FS3) For each $\alpha \in \Pi$, if there exists $\alpha' \in R$ such that $\pi_a(\alpha) = \pi_a(\alpha')$ and $2\alpha' \in R$, then $2\alpha \in R$.

It is clear that a fundamental-set $\Pi \cup \{a\}$ exists.

Let R and $\Pi \cup \{a\}$ be as in Definition 1.2, where $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_l\}$; in this paper, we always assume α_0 to be such that

$$(1.12) \quad \{\pi(\alpha_1), \dots, \pi(\alpha_l)\} \text{ is a base of } \pi(R)$$

(see also the last paragraph in the previous subsection). Let $\delta(\Pi) \in \mathbb{Z}\Pi$ be such that

$$(1.13) \quad \delta(\Pi) \in \mathbb{N}\Pi \text{ and } \mathbb{Z}\delta(\Pi) = \mathbb{Z}\Pi \cap V^0$$

(cf. Lemma 2.2 (2)). Then $\pi_a(\delta(\Pi)) = \delta(\pi_a(\Pi))$ (see (1.8) for $\delta(\pi_a(\Pi))$).

Let $\delta = \delta(\Pi)$ be of (1.13). By (1.9) and (1.6), for $\mathbb{X} = \mathbb{Z}$ or \mathbb{R} , we have

$$(1.14) \quad \begin{cases} \mathbb{X}R = \bigoplus_{\lambda \in \Pi \cup \{a\}} \mathbb{X}\lambda = \left(\bigoplus_{\alpha \in \Pi \setminus \{\alpha_0\}} \mathbb{X}\alpha \right) \oplus \mathbb{X}\delta \oplus \mathbb{X}a, \\ \mathbb{X}R \cap V^0 = \mathbb{X}\delta \oplus \mathbb{X}a, \\ R \subset (\mathbb{X}_+ \Pi \oplus \mathbb{X}a) \cup (\mathbb{X}_- \Pi \oplus \mathbb{X}a), \end{cases}$$

where $\mathbb{X}_\pm = \{x \in \mathbb{X} \mid \pm x \geq 0\}$.

Set

$$(1.15) \quad A(\Pi) := A(\pi_a(\Pi))$$

(see (1.10) for $A(\pi_a(\Pi))$).

1.4 Maps k and g and notations $c(\alpha)$, α^* and \mathcal{A}

In this subsection, we assume $l \geq 2$. We let R be an elliptic root system of rank l and we let $\Pi \cup \{a\}$ be a fundamental-set of R (see Definition 1.2). We can define maps

$$(1.16) \quad k : \Pi \rightarrow \{1, 2, 3, 4\} \text{ and } g : \Pi \rightarrow \{\emptyset, 2\mathbb{Z} + 1, \mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}\}$$

by

$$(1.17) \quad R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a) = \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha + \mathbb{Z}k(\alpha)a) \cup (2\varepsilon\alpha + g(\alpha)k(\alpha)a))$$

($\alpha \in \Pi$) (see (3.2); see also (1.14)). (If $g(\alpha) = \emptyset$, then $\alpha + g(\alpha)k(\alpha)a = \emptyset$.) Then we have an expression (1.18) of R below, which is a key to this paper:

$$(1.18) \quad R = \bigcup_{w \in W_\Pi} \left(\bigcup_{\alpha \in \Pi} ((w(\alpha) + \mathbb{Z}k(\alpha)a) \cup (w(2\alpha) + g(\alpha)k(\alpha)a)) \right)$$

(see also (3.3) (and (1.14))), where $W_\Pi = \langle s_\alpha | \alpha \in \Pi \rangle$ (see (1.1)). By (1.18), R is determined by Π , k and g . So

$$(1.19) \quad \text{we also denote } R \text{ by } R(\Pi, k, g).$$

(Caution: We redefine the notation $R(\Pi, k, g)$ in (3.15)).

Let (Π, k, g) be as above. Let $\alpha \in \Pi$. Let

$$(1.20) \quad c(\alpha) = \begin{cases} 2 & \text{if } g(\alpha) = \mathbb{Z} \text{ or } 2\mathbb{Z} + 1, \\ 1 & \text{if } g(\alpha) = \emptyset, 2\mathbb{Z} \text{ or } 4\mathbb{Z}, \end{cases}$$

and set

$$(1.21) \quad \alpha^* = c(\alpha)\alpha + k(\alpha)a \in R \text{ (see (1.17))}.$$

(In fact, $\{\alpha, -\alpha^*\}$ is a base of the rank-one affine root system $R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a)$ (see Remark 1.1 (1) and Lemma 4.3 (1)).) Recall $p(\cdot)$ from (1.2). By (1.17), we have:

$$(1.22) \quad p(\alpha) = p(-\alpha) = \begin{cases} 0 & \text{if } g(\alpha) = \emptyset \text{ or } 2\mathbb{Z} + 1, \\ 1 & \text{if } g(\alpha) = \mathbb{Z}, 2\mathbb{Z} \text{ or } 4\mathbb{Z}, \end{cases}$$

and

$$(1.23) \quad p(\alpha^*) = p(-\alpha^*) = \begin{cases} 0 & \text{if } g(\alpha) = \emptyset, \mathbb{Z}, 2\mathbb{Z} + 1 \text{ or } 4\mathbb{Z}, \\ 1 & \text{if } g(\alpha) = 2\mathbb{Z}. \end{cases}$$

We can define a subset \mathcal{A} of $\Pi \times \Pi \times \{1, 2, 3\}$ by

$$(1.24) \quad \mathcal{A} = \left\{ \left(\alpha, \beta, \frac{k(\beta)}{k(\alpha)} \right) \mid \alpha \neq \beta, (\alpha, \beta) \neq 0, (\alpha, \alpha) \leq (\beta, \beta) \right\}$$

($\frac{k(\beta)}{k(\alpha)} \in \{1, 2, 3\}$ follows from Theorem 3.1 (iii); see also Remark 1.1 (2) below).

Remark 1.1. Let (Π, k, g) be as above. (Recall $l \geq 2$.) Then we have:

(1) Let $\alpha \in \Pi$. Comparing (1.17) with (2.4), we see that $R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a)$ is a rank-one affine root system with the base $\{\alpha, -\alpha^*\}$ (see also Lemma 4.3 (1)).

(2) Let $(\alpha, \beta, y) \in \mathcal{A}$. Let

$$\lambda = \begin{cases} \beta & \text{if } y = c(\alpha) = 1, \\ \alpha & \text{otherwise.} \end{cases}$$

Comparing (3.11) with (2.6), we see that $R \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta \oplus \mathbb{R}a)$ is a rank-two affine root system. By (2.9), we see that there exists $w \in W_{\{\alpha, \beta\}}$ such that $\{\alpha, \beta, w(-\lambda^*)\}$ is a base of $R \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta \oplus \mathbb{R}a)$ (recall $W_{\{\alpha, \beta\}} = \langle s_\alpha, s_\beta \rangle$ (see (1.1))) (see also Lemma 4.5 (1)).

1.5 A Lie superalgebra and a super-symmetric invariant form

Let $\mathfrak{v} = \mathfrak{v}(0) \oplus \mathfrak{v}(1)$ be a $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -linear space; for $j \in \mathbb{Z}$, if $i \in \{0, 1\}$ is such that $j - i \in 2\mathbb{Z}$, then we also denote $\mathfrak{v}(i)$ by $\mathfrak{v}(j)$. If $X \in \mathfrak{v}(0)$ (resp. $X \in \mathfrak{v}(1)$), then we write

$$(1.25) \quad p(X) = 0 \text{ (resp. } p(X) = 1)$$

and we say that X is an *even* (resp. *odd*) element. If $X \in \mathfrak{v}(0) \cup \mathfrak{v}(1)$, then we say that X is a *homogeneous* element and that $p(X)$ is the *parity* of X . If a subspace \mathfrak{w} of \mathfrak{v} is such that $\mathfrak{w} = (\mathfrak{w} \cap \mathfrak{v}(0)) \oplus (\mathfrak{w} \cap \mathfrak{v}(1))$ (resp. $\mathfrak{w} \subset \mathfrak{v}(0)$, resp. $\mathfrak{w} \subset \mathfrak{v}(1)$), then we say that \mathfrak{w} is a *graded* (resp. *even*, resp. *odd*) subspace.

Let $\mathfrak{a} = \mathfrak{a}(0) \oplus \mathfrak{a}(1)$ be a $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{C} -linear space equipped with a bilinear map $[\cdot, \cdot] : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathbb{C}$ such that $[\mathfrak{a}(i), \mathfrak{a}(j)] \subset \mathfrak{a}(i+j)$ ($i, j \in \mathbb{Z}$); we recall from the above paragraph that

$$(1.26) \quad \mathfrak{a}(i) = \{X \in \mathfrak{a} \mid p(X) = i\}.$$

We say that $\mathfrak{a} = (\mathfrak{a}, [\cdot, \cdot])$ is a (\mathbb{C}) -Lie superalgebra if it satisfies that for all homogeneous elements X, Y, Z of \mathfrak{a} ,

$$\begin{cases} [Y, X] = -(-1)^{p(X)p(Y)}[X, Y], \\ [X, [Y, Z]] = [[X, Y], Z] + (-1)^{p(X)p(Y)}[Y, [X, Z]] \end{cases}$$

(the second equality is called the *Jacobi identity*). For $X \in \mathfrak{a}$, we define $\text{ad}X \in \text{End}_{\mathbb{C}}(\mathfrak{a})$ by $(\text{ad}X)(Y) = [X, Y]$.

We say that a bilinear form $(,) : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathbb{C}$ is a *super-symmetric invariant form* on \mathfrak{a} if it satisfies that for all homogeneous elements X, Y, Z of \mathfrak{a} ,

$$(Y, X) = (-1)^{p(X)p(Y)}(X, Y) \text{ and } (X, [Y, Z]) = ([X, Y], Z).$$

1.6 Definition of the elliptic Lie superalgebras $\mathfrak{g}(\Pi, k, g)$ of rank $l \geq 2$

Let $R = R(\Pi, k, g)$ be an elliptic root system of rank $l \geq 2$ (see (1.19)), where $\Pi = \{\alpha_0, \dots, \alpha_l\}$. Let $\mathcal{E} = \mathcal{E}(\Pi, k, g)$ be the $(l+4)$ -dimensional \mathbb{C} -linear space with a basis $\{\alpha_0, \dots, \alpha_l, a, \Lambda_\delta, \Lambda_a\}$. We identify $V = \mathbb{R}R(\Pi, k, g)$ with the \mathbb{R} -linear subspace of \mathcal{E} spanned by $\alpha_0, \dots, \alpha_l, a$. Let $\delta = \delta(\Pi) (\in \mathbb{N}\Pi \cap V^0)$ be of (1.13). As for α_0 , we recall (1.12). Then $\{\alpha_1, \dots, \alpha_l, \delta, a, \Lambda_\delta, \Lambda_a\}$ is also a basis of \mathcal{E} (cf. (1.14)), i.e.,

$$\begin{aligned} (1.27) \quad \mathcal{E} &= \mathcal{E}(\Pi, k, g) \\ &= \left(\bigoplus_{i=0}^l \mathbb{C}\alpha_i \right) \oplus \mathbb{C}a \oplus \mathbb{C}\Lambda_\delta \oplus \mathbb{C}\Lambda_a \\ &= \left(\bigoplus_{i=1}^l \mathbb{C}\alpha_i \right) \oplus \mathbb{C}\delta \oplus \mathbb{C}a \oplus \mathbb{C}\Lambda_\delta \oplus \mathbb{C}\Lambda_a. \end{aligned}$$

We extend $(,)$ on V to the non-degenerate symmetric bilinear form on \mathcal{E} by $(\delta, \Lambda_\delta) = (a, \Lambda_a) = 1$, $(\Lambda_\delta, \Lambda_\delta) = (\Lambda_a, \Lambda_a) = (\Lambda_\delta, \Lambda_a) = 0$, $(\Lambda_a, \delta) = (\Lambda_\delta, a) = 0$ and $(\Lambda_\delta, \alpha) = (\Lambda_a, \alpha) = 0$ ($\alpha \in \Pi \setminus \{\alpha_0\}$). Set

$$(1.28) \quad \mathcal{B}_+ := \{\alpha, \alpha^* | \alpha \in \Pi\} \quad \text{and} \quad \mathcal{B} := \mathcal{B}_+ \cup (-\mathcal{B}_+).$$

For $\mu, \nu \in \mathcal{B}$ with $\mu \neq \nu$ and $\mu + \nu \neq 0$, define $x_{\mu, \nu} \in \mathbb{N}$ by

$$(1.29) \quad x_{\mu, \nu} = \begin{cases} 1 - (\mu^\vee, \nu) & \text{if } (\mu^\vee, \nu) < 0, \\ 1 & \text{if } (\mu^\vee, \nu) \geq 0. \end{cases}$$

Definition 1.3. (*Definition of $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$*) Let $R = R(\Pi, k, g)$ (see (1.19)) be an elliptic root system of rank $l \geq 2$. Let $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$ ($= \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ (cf. (1.26))) be the \mathbb{C} -Lie superalgebra defined by generators:

$$h_\sigma \ (\sigma \in \mathcal{E}), \quad E_\mu \ (\mu \in \mathcal{B})$$

with parities:

$$p(h_\sigma) = 0, \quad p(E_\mu) = p(\mu)$$

and defining relations:

- (SR1) $xh_\sigma + yh_\tau = h_{x\sigma+y\tau}$ if $x, y \in \mathbb{C}$ and $\sigma, \tau \in \mathcal{E}$,
- (SR2) $[h_\sigma, h_\tau] = 0$ if $\sigma, \tau \in \mathcal{E}$,
- (SR3) $[h_\sigma, E_\mu] = (\sigma, \mu)E_\mu$ if $\sigma \in \mathcal{E}$ and $\mu \in \mathcal{B}$,
- (SR4) $[E_\mu, E_{-\mu}] = h_{\mu^\vee}$ if $\mu \in \mathcal{B}_+$,
- (SR5) $(\text{ad}E_\mu)^{x_{\mu,\nu}}E_\nu = 0$ if $\mu, \nu \in \mathcal{B}$ are such that $\mu \neq \nu$ and $\mu + \nu \neq 0$,
- (SR6) $c(\alpha)(\text{ad}E_{\alpha^*})^y E_\beta = (\text{ad}E_\alpha)^{c(\alpha)y} E_{\beta^*}$ if $(\alpha, \beta, y) \in \mathcal{A}$,
- (SR7) $(-1)^{c(\alpha)+1} c(\alpha)(\text{ad}E_{-\alpha^*})^y E_{-\beta} = (\text{ad}E_{-\alpha})^{c(\alpha)y} E_{-\beta^*}$ if $(\alpha, \beta, y) \in \mathcal{A}$,
- (SR8) $(\text{ad}E_\alpha)^i (\text{ad}E_{\alpha^*})^{y-i} E_\beta = 0$ if $(\alpha, \beta, y) \in \mathcal{A}$ and $1 \leq i \leq y-1$,
- (SR9) $(\text{ad}E_{-\alpha})^i (\text{ad}E_{-\alpha^*})^{y-i} E_{-\beta} = 0$ if $(\alpha, \beta, y) \in \mathcal{A}$ and $1 \leq i \leq y-1$

(see (1.20)-(1.29) for the notation).

Let $\mathfrak{h} = \mathfrak{h}(\Pi, k, g)$ be the \mathbb{C} -subspace of \mathfrak{g} formed by the elements h_σ ($\sigma \in \mathcal{E}$). For $\sigma \in \mathcal{E}$, let $\mathfrak{g}_\sigma = \{X \in \mathfrak{g} \mid [h_\lambda, X] = (\sigma, \lambda)X \ (\lambda \in \mathcal{E})\}$, so $\mathfrak{g}_\sigma = (\mathfrak{g}_\sigma \cap \mathfrak{g}(0)) \oplus (\mathfrak{g}_\sigma \cap \mathfrak{g}(1))$ (cf. (1.26)).

See (4.18) (resp. (4.19), resp. (7.5)) for a significance of (SR6,7) (resp. (SR8,9), resp. (SR5)).

(Caution: We redefine the notation $\mathfrak{g}(\Pi, k, g)$ in (3.26)).

By (SR1), we can consider the number of the defining relations in (SR1-9) to be finite.

1.7 Main theorem

Now we state our main theorem.

Theorem 1.1. *Let (R, V) be an elliptic root system of rank $l \geq 2$. Let $\Pi \cup \{a\}$ be a fundamental-set of R (see Definition 1.2). Define maps $k : \Pi \rightarrow \{1, 2, 3, 4\}$ and $g : \Pi \rightarrow \{\emptyset, 2\mathbb{Z} + 1, \mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z}\}$ by (1.17) (see also (3.2)) (so $R = R(\Pi, k, g)$ (see (1.19))). Assume that $g(\alpha) \neq \emptyset$ for some $\alpha \in \Pi$. Let $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$ and $\mathfrak{h} = \mathfrak{h}(\Pi, k, g)$ be of Definition 1.3. Let $\mathcal{E} = \mathcal{E}(\Pi, k, g)$ be of (1.27). Let $\delta = \delta(\Pi)$ be of (1.13). Then we have the following:*

(1) \mathfrak{g} has the properties (1.30)-(1.32) below:

$$(1.30) \quad \mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\mu \in R} \mathfrak{g}_\mu \right) \oplus \left(\bigoplus_{(m,r) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \mathfrak{g}_{m\delta+ra} \right)$$

$$(1.31) \quad \dim \mathfrak{h} = l + 4, \text{ i.e., } h_\sigma \neq 0 \text{ for all } \sigma \in \mathcal{E} \setminus \{0\}$$

$$(1.32) \quad \dim \mathfrak{g}_\mu = 1 \text{ for all } \mu \in R = R(\Pi, k, g)$$

Moreover, $\mathfrak{g}_\mu \subset \mathfrak{g}(p(\mu))$ for $\mu \in R$ (see (1.2) for $p(\mu)$ and see (1.26) for $\mathfrak{g}(p(\mu))$).

(2) There exists a super-symmetric invariant form $(,) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ having the properties below:

$$(1.33) \quad (h_\sigma, h_\tau) = (\sigma, \tau) \quad (\sigma, \tau \in \mathcal{E})$$

$$(1.34) \quad \ker(,) \subset \bigoplus_{(m,r) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \mathfrak{g}_{m\delta+ra}$$

(3) Let \mathfrak{g}' be a \mathbb{C} -Lie superalgebra satisfying the following conditions.

(3-i) \mathfrak{g}' includes \mathfrak{h} as a sub-Lie superalgebra (more precisely, $\mathfrak{h} \subset \mathfrak{g}'(0)$ (see (1.26) for the notation $\mathfrak{g}'(0)$)).

(3-ii) \mathfrak{g}' has the same properties as those in (1.30)-(1.32) with \mathfrak{g}' in place of \mathfrak{g} .

(3-iii) \mathfrak{g}' has a super-symmetric invariant form having the same properties as those in (1.33)-(1.34).

(3-iv) \mathfrak{g}' is generated by \mathfrak{h} and \mathfrak{g}'_μ with all $\mu \in R$, where, for $\sigma \in \mathcal{E}$, we let $\mathfrak{g}'_\sigma = \{X \in \mathfrak{g}' \mid [h_\lambda, X] = (\sigma, \lambda)X \ (\lambda \in \mathcal{E})\}$.

Then there exists an epimorphism $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ such that $f(h_\sigma) = h_\sigma$ for all $\sigma \in \mathcal{E}$ (this implies $f(\mathfrak{g}_\sigma) = \mathfrak{g}'_\sigma$ ($\sigma \in \mathcal{E}$)).

Proof of the theorem shall be given in Subsections 4.9, 6.2 and 7.2. For the strategy of the proof, see Subsection 1.8.

Remark 1.2. (1) A result similar to Theorem 1.1 for g such that $g(\alpha) = \emptyset$ for all $\alpha \in \Pi$ has been given in [18].

(2) By Theorem 3.1 and Lemma 3.1 (2), we shall see that $R(\Pi, k, g)$ is reduced if and only if $g(\Pi) \subset \{\emptyset, 2\mathbb{Z} + 1\}$. There exist two isomorphism classes of the reduced elliptic root systems $R(\Pi, k, g)$ of rank $l \geq 2$ which are not isomorphic to $R(\Pi', k', g')$ for any (Π', k', g') such that $g'(\Pi) = \{\emptyset\}$ (see [1] and [2]).

(3) If $R = R(\Pi, k, g)$ is of Theorem 1.1 and $l \geq 3$ (resp. $l = 2$), then $\pi(R)$ is B_l (resp. C_2) and $A(\Pi)$ is the same as the one in (3.29). Hence, by (3.13), we shall see $3 \notin k(\Pi)$. We shall also see that if $3 \in k(\Pi)$, then $\pi(R)$ is G_2 . (See also Subsection 1.9 for the terminology.)

(4) See [3] for the classification of the non-reduced elliptic root systems.

1.8 Strategy of Proof of Theorem 1.1

In this subsection, we explain about strategy of Proof of Theorem 1.1.

Strategy of Proof of Theorem 1.1 (1) (See Subsection 4.9 for the proof) A prototype of our proof is the one of the Gabber-Kac theorem [7], [8, Theorem 9.11], which states that the (symmetrizable) Kac-Moody Lie algebras \mathfrak{KM} can also be defined by the generators, called the *Chevalley generators*, and the relations, called the *Serre relations*. (Namely, it is the Serre theorem for \mathfrak{KM} .) Outline of the proof of the Gabber-Kac theorem is as follows: Let \mathfrak{KM}' be the Lie algebra defined by the Chevalley generators and the Serre relations. Let $f : \mathfrak{KM}' \rightarrow \mathfrak{KM}$ be the natural epimorphism. It follows that \mathfrak{KM}' , as well as \mathfrak{KM} , has the triangular decomposition and has the property that the Weyl group action on the set of the roots preserves their multiplicities. Then \mathfrak{KM}' and \mathfrak{KM} have the properties:

$$(1.35) \quad \left\{ \begin{array}{l} \text{(i) The dimension of the Cartan subalgebra (i.e., the zero} \\ \text{root space) is } \text{rank} \mathfrak{A} + 2 \dim \ker \mathfrak{A}, \text{ where } \mathfrak{A} \text{ is the generalized} \\ \text{Cartan matrix for which } \mathfrak{KM} \text{ and } \mathfrak{KM}' \text{ are defined.} \\ \text{(ii) The set of the (non-zero) roots are formed by} \\ \text{the real and imaginary roots.} \\ \text{(iii) The multiplicity of any real root is one.} \end{array} \right.$$

(cf. [8, Chapter 5]), where a root is called *real* if it is conjugate to a simple root under the Weyl group action; otherwise, it is called *imaginary*. By (1.35), $\ker f$ is a subspace of the direct sum of the imaginary root spaces.

The proof is completed by showing that $\ker f$ is really $\{0\}$ (this is done by using a technical fact [8, Proposition 9.11]).

Strictly speaking, our prototype is the proof of (1.35). We notice that (1.35) is similar to Lemma 4.2 in this paper.

In the proof of Theorem 1.1 (1), we use the triangular decomposition:

$$(1.36) \quad \mathfrak{g} = \left(\bigoplus_{\mu \in (\mathbb{Z}_+ \Pi \oplus \mathbb{Z}a) \setminus \mathbb{Z}a} \mathfrak{g}_\mu \right) \bigoplus (\mathfrak{l}^+ \oplus \mathfrak{h} \oplus \mathfrak{l}^-) \bigoplus \left(\bigoplus_{\nu \in (\mathbb{Z}_- \Pi \oplus \mathbb{Z}a) \setminus \mathbb{Z}a} \mathfrak{g}_\nu \right)$$

(see (1.42)), where $\mathfrak{l}^\pm = \bigoplus_{m \in \pm \mathbb{N}} \mathfrak{g}_{ma}$, and we also use the fact:

$$(1.37) \quad \dim \mathfrak{g}_{w(\sigma)} = \dim \mathfrak{g}_\sigma \quad \text{for } w \in W_\Pi \text{ and } \sigma \in \mathcal{E}$$

(see (4.11)), where $W_\Pi = \langle s_\alpha | \alpha \in \Pi \rangle$ (see (1.1)).

Let us explain the strategy more precisely. (*Caution:* As a matter of fact, we get Theorem 1.1 (1) as an immediate consequence of Lemma 4.10, which is a wider-version of Theorem 1.1 (1). However, by Theorem 3.2, we see that Lemma 4.10 is not properly wider than Theorem 1.1 (1).)

To each \mathfrak{g} , we attach a Lie superalgebra \mathbb{L} and a homomorphism $\psi_i : \mathfrak{g} \rightarrow \mathbb{L}$ such that $\dim \psi_i(\mathfrak{h}) = l + 4$ (see Lemmas 4.1, 5.5-5.9 and Subsections 5.1 and 5.9; Subsection 5.1 is strategy). Thus we have (1.31).

We use notation R^S and \mathfrak{g}^S as follows. Let S be a non-empty subset of Π . Set $R^S := (\mathbb{R}S \oplus \mathbb{R}a) \cap R$ (cf. (4.24)), so $R^S = (\mathbb{Z}S \oplus \mathbb{Z}a) \cap R$ (cf. (1.14)). Let $R^{S,+} = (\mathbb{Z}_+ S \oplus \mathbb{Z}a) \cap R$ and $R^{S,-} = (\mathbb{Z}_- S \oplus \mathbb{Z}a) \cap R$ (cf. (4.59)). By (1.14), we have

$$(1.38) \quad R^S = R^{S,+} \cup R^{S,-}$$

(cf. (4.60)). Let \mathfrak{g}^S (cf. (4.25)) be the sub-Lie superalgebra of \mathfrak{g} generated by \mathfrak{h} and $E_\mu, E_{-\mu}, E_{\mu^*}, E_{-\mu^*}$ with all $\mu \in S$. For $\sigma \in \mathcal{E}$, let $\mathfrak{g}_\sigma^S = \mathfrak{g}^S \cap \mathfrak{g}_\sigma$.

If $|S| \leq 2$, then (by Lemmas 4.3, 4.5 and 4.6) we have

$$(1.39) \quad \mathfrak{g}^S = \mathfrak{h} \bigoplus \left(\bigoplus_{\mu \in R^S} \mathfrak{g}_\mu^S \right) \bigoplus \left(\bigoplus_{\lambda \in (\mathbb{Z} \setminus \{0\})k_S a} \mathfrak{g}_\lambda^S \right),$$

where $k_S = \min k(S)$, and we have

$$(1.40) \quad \dim \mathfrak{g}_\mu^S = 1 \quad \text{for all } \mu \in R^S.$$

(*Strategy of the proof of (1.39)-(1.40).* We use Lemma 4.2 in this paper, which is a weaker version of (the super-version of) the Gabber-Kac theorem [7].

Case 1: $|S| = 1$ and $S = \{\alpha\}$. By (1.17) (or (3.2)), we can see that $R^{\{\alpha\}}$ is an affine root system with a base $\{\alpha, -\alpha^*\}$ (see Lemma 4.3 (1)). Since $E_{\pm\alpha}, E_{\pm\alpha^*}$ satisfy the Serre relations (see (SR1-5)), we have (1.39)-(1.40) (see Lemma 4.3 (2)) by using Lemma 4.2 (and (1.31)).

Case 2: $|S| = 2$ and $S = \{\alpha, \beta\}$ with $(\alpha, \beta) \neq 0$. Assume that $k(\alpha) < k(\beta)$ or $k(\alpha) = k(\beta)$ and $(\alpha^*, \alpha^*) \geq (\beta^*, \beta^*)$ (cf. (4.15)). Let $x = -(\beta^\vee, \alpha^*) \in \{1, 2\}$ and $\gamma = -(\alpha^* + x\beta) (= -s_\beta(\alpha^*))$. Since $R^{\{\alpha, \beta\}} = W_{\{\alpha, \beta\}} \cdot (R^{\{\alpha\}} \cup R^{\{\beta\}})$ (more precisely, (3.11)), we can see that $R^{\{\alpha, \beta\}}$ is an affine root system with a base $\{\alpha, \beta, \gamma\}$. Let

$$E_{\pm\gamma} := (\pm 1)^{p(\alpha^*)} (\mp 1)^x x^{-1} (\text{ad} E_{\mp\beta})^x E_{\mp\alpha^*}$$

(see (4.42)). We show by case-by-case checking that h_σ ($\sigma \in \mathcal{E}$) and E_μ ($\mu \in \{\pm\alpha, \pm\beta, \pm\gamma\}$) generate $\mathfrak{g}^{\{\alpha, \beta\}}$ and satisfy the Serre relations (see Step 4 of Proof of Lemma 4.5 (2) (and (4.23))). Then we have (1.39)-(1.40) (see Lemma 4.5 (2)) by using Lemma 4.2 (and (1.31)).

Case 3: $|S| = 2$ and $S = \{\alpha, \beta\}$ with $(\alpha, \beta) = 0$. Then (1.39)-(1.40) are clear from Case 1 since $\mathfrak{g}^{\{\alpha, \beta\}} = \mathfrak{g}^{\{\alpha\}} + \mathfrak{g}^{\{\beta\}}$ and $R^{\{\alpha, \beta\}} = R^{\{\alpha\}} \cup R^{\{\beta\}}$ (see (4.56)-(4.57)).

Let $\alpha, \beta \in \Pi$ be such that $\alpha \neq \beta$. Since $\mathfrak{g}^{\{\alpha\}}, \mathfrak{g}^{\{\beta\}} \subset \mathfrak{g}^{\{\alpha, \beta\}}$, by (1.38), (1.39) and (1.40), we have

$$(1.41) \quad \begin{cases} [\mathfrak{g}_\mu^{\{\alpha\}}, \mathfrak{g}_\nu^{\{\beta\}}] = \{0\} & \text{for } \mu \in R^{\{\alpha\}, +} \text{ and } \nu \in R^{\{\beta\}, -}, \\ [\mathfrak{g}_\mu^{\{\alpha\}}, \mathfrak{g}_\lambda^{\{\beta\}}] \subset \mathfrak{g}_{\mu+\lambda}^{\{\alpha\}} & \text{for } \mu \in R^{\{\alpha\}} \text{ and } \lambda \in \mathbb{Z}k(\beta)a, \end{cases}$$

where if $\mu + \lambda \notin R^{\{\alpha\}}$, $\mathfrak{g}_{\mu+\lambda}^{\{\alpha\}} = \{0\}$ (see (4.64)). Since \mathfrak{g} is generated by \mathfrak{h} and $\mathfrak{g}_\mu^{\{\alpha\}}$ with all $\alpha \in \Pi$ and $\mu \in R^{\{\alpha\}}$, by (1.41) (and (1.31) and (1.38)-(1.39) for $S = \{\alpha\}$ with all $\alpha \in \Pi$), we have

$$(1.42) \quad \mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{l}^+ \oplus \mathfrak{h} \oplus \mathfrak{l}^- \oplus \mathfrak{n}^-$$

(see (4.61)), where we let \mathfrak{n}^\pm be the sub-Lie superalgebras of \mathfrak{g} generated by $\mathfrak{g}_\mu^{\{\alpha\}}$ with all $\alpha \in \Pi$ and all $\mu \in R^{\{\alpha\}, \pm}$ and we let $\mathfrak{l}^\pm = \sum_{\alpha \in \Pi} \sum_{m \in \mathbb{N}} \mathfrak{g}_{\pm mk(\alpha)a}^{\{\alpha\}}$. By (1.42), we have (1.36) (more precisely, $\mathfrak{n}^\pm = \bigoplus_{\mu \in (\mathbb{Z}_\pm \Pi \oplus \mathbb{Z}a) \setminus \mathbb{Z}a} \mathfrak{g}_\mu$ and $\mathfrak{l}^\pm = \bigoplus_{m \in \pm \mathbb{N}} \mathfrak{g}_{ma}$). Let $\alpha \in \Pi$. By the definition of \mathfrak{n}^\pm , we have $\mathfrak{n}^\pm \cap (\bigoplus_{\lambda \in \mathbb{Z}\alpha \oplus \mathbb{Z}a} \mathfrak{g}_\lambda) = \bigoplus_{\lambda \in R^{\{\alpha\}, \pm}} \mathfrak{g}_\lambda^{\{\alpha\}}$. Then, by (1.40) and (1.42), we have

$$(1.43) \quad \dim \mathfrak{g}_\lambda = \begin{cases} 1 & \text{if } \lambda \in R^{\{\alpha\}}, \\ 0 & \text{if } \lambda \in (\mathbb{Z}\alpha \oplus \mathbb{Z}a) \setminus (R^{\{\alpha\}} \cup \mathbb{Z}a) \end{cases}$$

(cf. (4.70) and (4.71)).

Since $E_{\pm\alpha}$ ($\alpha \in \Pi$) are locally ad-nilpotent (see Subsection 4.2 for the terminology (cf. (4.5))), we have (1.37) (see (4.11)). We can prove

$$(1.44) \quad \left\{ \begin{array}{l} R = W_{\Pi} \cdot (\cup_{\alpha \in \Pi} R^{\{\alpha\}}) \text{ (see (4.32)) and} \\ (\mathbb{Z}\Pi \oplus \mathbb{Z}a) \setminus (R \cup (\mathbb{Z}\delta \oplus \mathbb{Z}a)) \subset \\ W_{\Pi} \cdot \left((\cup_{\alpha \in \Pi} ((\mathbb{Z}\alpha \oplus \mathbb{Z}a) \setminus (R^{\{\alpha\}} \cup \mathbb{Z}a))) \right. \\ \left. \cup ((\mathbb{Z}\Pi \oplus \mathbb{Z}a) \setminus ((\mathbb{Z}_+\Pi \oplus \mathbb{Z}a) \cup (\mathbb{Z}_-\Pi \oplus \mathbb{Z}a))) \right) \text{ (see (4.33)).} \end{array} \right.$$

By (1.36), (1.37), (1.43) and (1.44), we have (1.30) and (1.32), as desired.

Strategy of Proof of Theorem 1.1 (2) (See Subsection 6.2 for the proof) Using a super-symmetric invariant form on \mathbb{L} (see above for \mathbb{L}), we have a super-symmetric invariant form $(,)'$ on \mathfrak{g} (see (6.11)). Then (using Lemma 6.1) we get $(,)$ by normalizing $(,)'$.

Strategy of Proof of Theorem 1.1 (3) (See Subsection 7.2 for the proof) Using (3-i)-(3-iii) (and Lemma 7.1), we see that for each $\mu \in \mathcal{B}_+$, if $p(\mu) = 0$ (resp. $p(\mu) = 1$), then $\mathbb{C}h_{\mu^\vee} \oplus (\oplus_{m=1}^{p(\mu)+1} (\mathfrak{g}'_{m\mu} \oplus \mathfrak{g}'_{-m\mu}))$ is a sub-Lie superalgebra of \mathfrak{g}' isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ (resp. $\mathfrak{osp}(2|1)$). In particular, $\mathfrak{g}'_{\pm\mu} \subset \mathfrak{g}'(p(\mu))$ and there exist $E'_{\pm\mu} \in \mathfrak{g}'_{\pm\mu}$ such that $[E'_{\mu}, E'_{-\mu}] = h_{\mu^\vee}$ (and $\mathbb{C}E'_{\pm\mu} = \mathfrak{g}'_{\pm\mu}$). Hence h_{σ} ($\sigma \in \mathcal{E}$) and $E'_{\pm\mu}$ ($\mu \in \mathcal{B}_+$) satisfy (SR1-4). They also satisfy (SR5,8,9), since the corresponding root spaces do not exist (see (4.19) and (7.5)). Using (3-ii), for $(\alpha, \beta, y) \in \mathcal{A}$, we can show (instead of (SR6,7)) that

$$(\pm 1)^{c(\alpha)+1} c(\alpha) (\text{ad} E'_{\pm\alpha^*})^y E'_{\pm\beta} = c_{\alpha,\beta}^{\pm 1} (\text{ad} E'_{\pm\alpha})^{c(\alpha)y} E'_{\pm\beta^*}$$

for some $c_{\alpha,\beta} \in \mathbb{C} \setminus \{0\}$ (see (7.8)). However, since $g(\alpha) \neq \emptyset$ for some $\alpha \in \Pi$, $A(\Pi)$ (see (1.15)) is not $A_l^{(1)}$, i.e., the Dynkin diagram associated with $A(\Pi)$ is not a loop. So we see that there exist $z_{\mu} \in \mathbb{C} \setminus \{0\}$ ($\mu \in \mathcal{B}_+$) such that $z_{\mu}^{\pm 1} E'_{\pm\mu}$ satisfy (SR6,7). Thus we can define a homomorphism $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ by $f(h_{\sigma}) = h_{\sigma}$ and $f(E'_{\pm\mu}) = z_{\mu}^{\pm 1} E'_{\pm\mu}$. Using (3-iv) (and Lemma 4.4 (3)) and using automorphisms n'_{μ} of \mathfrak{g}' ($\mu \in \mathcal{B}_+$) having the properties $n'_{\mu}(\mathfrak{g}'_{\sigma}) = \mathfrak{g}'_{s_{\mu}(\sigma)}$ and $n'_{\mu}(\text{Im} f) = \text{Im} f$, we can see that f is surjective.

1.9 Notation R_{sh} , R_{lg} and R_{ex}

Here we recall the names of the irreducible finite root systems and the generalized Cartan matrices of affine-type. If R is an irreducible finite root system, then R is one of the following: A_l ($l \geq 1$), B_l ($l \geq 3$), C_l ($l \geq 2$), D_l ($l \geq 4$),

E_l ($l = 6, 7, 8$), G_2 ($l = 2$) and BC_l ($l \geq 1$) (see [4, Chapter VI §1 Definition 1 (SR_1)-(SR_3)] and [1]). If $l = 2$, we use A_2 , C_2 or G_2 ; we don't use B_2 .

Recall $A(\Pi)$ from (1.10). If R is an affine root system and Π is a base of R , then $A(\Pi)$ is one of the following:

$$(1.45) \quad \begin{cases} A_l^{(1)} \ (l \geq 1), B_l^{(1)} \ (l \geq 3), C_l^{(1)} \ (l \geq 2), D_l^{(1)} \ (l \geq 4), \\ E_l^{(1)} \ (l = 6, 7, 8), F_4^{(1)} \ (l = 4), G_2^{(1)} \ (l = 2), \\ A_{2l}^{(2)} \ (l \geq 1), A_{2l-1}^{(2)} \ (l \geq 3), D_{l+1}^{(2)} \ (l \geq 2), E_6^{(2)} \ (l = 4), \\ D_4^{(3)} \ (l = 2) \end{cases}$$

(see [8]; see also Section 9).

Let R be an extended affine root system. Define subsets R_{sh} , R_{lg} and R_{ex} of R by

$$R_{\text{sh}} = \{\alpha \in R \mid (\alpha, \alpha) \leq (\beta, \beta) \text{ for all } \beta \in R\},$$

$R_{\text{ex}} = R \cap \pi^{-1}(2\pi(R_{\text{sh}}))$ and $R_{\text{lg}} = R \setminus (R_{\text{sh}} \cup R_{\text{ex}})$ (see (1.5) for π). We have

$$(1.46) \quad R = R_{\text{sh}} \cup R_{\text{lg}} \cup R_{\text{ex}} \text{ (disjoint union)}.$$

Notice that $R_{\text{ex}} = \emptyset$ if and only if the irreducible finite root system $\pi(R)$ is reduced. For a subset S of R , set

$$(1.47) \quad S_{\text{sh}} = S \cap R_{\text{sh}}, S_{\text{lg}} = S \cap R_{\text{lg}}, S_{\text{ex}} = S \cap R_{\text{ex}}.$$

2 Facts of the affine root systems

2.1 Bases of the affine root systems

The following lemma is well-known, but we state it for later use.

Lemma 2.1. *Let R be an irreducible finite root system. Let Π be a base of R (cf. (1.6)). Then we have the following:*

(1) $W_{\Pi} = W$ (see (1.1) for W_{Π} and see Definition 1.1 for $W = W_R$).

(2) For $\mu, \nu \in R$ with $(\mu, \mu) = (\nu, \nu)$, there exists $w \in W_{\Pi}$ such that $w(\mu) = \nu$.

(3) There exists a unique $\theta_{\text{sh}} = \theta_{\text{sh}}(R, \Pi) \in R$ such that $W_{\Pi} \cdot \theta_{\text{sh}} = R_{\text{sh}} \subset \theta_{\text{sh}} + \mathbb{Z}_- \Pi$. Moreover, if $R_{\text{lg}} \neq \emptyset$ (resp. $R_{\text{ex}} \neq \emptyset$), then there exists a unique $\theta_{\text{lg}} = \theta_{\text{lg}}(R, \Pi)$ (resp. $\theta_{\text{ex}} = \theta_{\text{ex}}(R, \Pi)$) $\in R$ such that $W_{\Pi} \cdot \theta_{\text{lg}} = R_{\text{lg}} \subset \theta_{\text{lg}} + \mathbb{Z}_- \Pi$ (resp. $W_{\Pi} \cdot \theta_{\text{ex}} = R_{\text{ex}} \subset \theta_{\text{ex}} + \mathbb{Z}_- \Pi$). Furthermore, $\theta_{\text{sh}} \in \text{NII}$, $\theta_{\text{lg}} \in \text{NII}$ (if $R_{\text{lg}} \neq \emptyset$) and $\theta_{\text{ex}} = 2\theta_{\text{sh}}$ (if $R_{\text{ex}} \neq \emptyset$).

Proof of this lemma shall be given in Section 8.

Keep the notation as in Lemma 2.1. Let $\Theta_{\text{sh}} = \{\theta_{\text{sh}}\}$. Let $\Theta_{\text{lg}} = \{\theta_{\text{lg}}\}$ (resp. $\Theta_{\text{lg}} = \emptyset$) if $R_{\text{lg}} \neq \emptyset$ (resp. $R_{\text{lg}} = \emptyset$). Let $\Theta_{\text{ex}} = \{\theta_{\text{ex}}\}$ (resp. $\Theta_{\text{ex}} = \emptyset$) if $R_{\text{ex}} \neq \emptyset$ (resp. $R_{\text{ex}} = \emptyset$). Set

$$(2.1) \quad \Theta(R, \Pi) := \Theta_{\text{sh}} \cup \Theta_{\text{lg}} \cup \Theta_{\text{ex}}.$$

Recall the definition of the generalized Cartan matrix of affine-type from [8, §4.3 and Proposition 4.7]. By [8, Theorem 4.8], we have the following.

Lemma 2.2. *Let $l \in \mathbb{N}$.*

(1) *Let $A = (a_{ij})_{0 \leq i, j \leq l}$ be an $(l+1) \times (l+1)$ -matrix. Then A is a generalized Cartan matrix of affine-type if and only if there exist a rank- l irreducible finite root system R with a base $\{\gamma_1, \dots, \gamma_l\}$ and an element γ_0 of $-\Theta(R, \Pi)$ (see (2.1)) such that $a_{ij} = (\gamma_i^\vee, \gamma_j)$ ($0 \leq i, j \leq l$), i.e., $A = A(\Pi)$ (cf. (1.10)), where we let $\Pi = \{\gamma_0, \dots, \gamma_l\}$. (See also (1.45) and Section 9.)*

(2) *Keep the notation as in Subsection 1.1. Assume $\dim \mathbb{V} = l+1$ and let $\{v_i | 0 \leq i \leq l\}$ be a basis of \mathbb{V} such that $v_i \in \mathbb{V}^\times$ and the $(l+1) \times (l+1)$ -matrix $((v_i^\vee, v_j))_{0 \leq i, j \leq l}$ is a generalized Cartan matrix of affine type. Then there exists a unique $x \in \bigoplus_{i=0}^l \mathbb{N}v_i$ such that $\mathbb{Z}x = \mathbb{V}^0 \cap (\bigoplus_{i=0}^l \mathbb{Z}v_i)$.*

The following theorem seems to be well-known (see [11]), but we state and prove it for later use.

Theorem 2.1. (cf. [11]) *Let $l \in \mathbb{N}$.*

(1) *Let (R, V) be an affine root system of rank l . Let δ' be a basis element of the rank-one \mathbb{Z} -module $\mathbb{Z}R \cap V^0$, i.e., $\mathbb{Z}\delta' = \mathbb{Z}R \cap V^0$ (cf. (1.4)). Let $\Pi' = \{\alpha_1, \dots, \alpha_l\}$ be a subset of R such that $\pi(\Pi')$ is a base of the irreducible finite root system $(\pi(R), V/\mathbb{R}\delta')$ (cf. (1.6) and (1.5)). (So $\mathbb{Z}R = (\bigoplus_{i=1}^l \mathbb{Z}\alpha_i) \oplus \mathbb{Z}\delta'$ (cf. (1.9)).) Then we have:*

(1-1) *There exists a unique $\alpha_0 = \alpha_0(R, \Pi', \delta') \in R$ such that $\{\alpha_0\} \cup \Pi'$ is a base of R and $\alpha_0 \in \mathbb{N}\delta' \oplus (\bigoplus_{i=1}^l \mathbb{Z}\alpha_i)$.*

(1-2) *$\alpha_0 = \delta' - \theta$, where $\theta \in \mathbb{N}\Pi'$ and $\pi(\theta) \in \Theta(\pi(R), \pi(\Pi'))$ (see (2.1)).*

(1-3) *$((\alpha_i^\vee, \alpha_j))_{0 \leq i, j \leq l}$ is a generalized Cartan matrix of affine-type.*

(1-4) *For $0 \leq i \leq l$, if $2\alpha_i \in R$, then $-(\alpha_i^\vee, \alpha_j) \in \{0, 2, 4\}$ for all $j \in \{0, \dots, l\} \setminus \{i\}$.*

(2) Let $A = (a_{ij})_{0 \leq i, j \leq l}$ be a generalized Cartan matrix of affine-type. Let

$$\bar{\tau} = \{i \in \{0, \dots, l\} \mid a_{ij} \in 2\mathbb{Z}_- \text{ for all } j \in \{0, \dots, l\} \setminus \{i\}\}$$

(so $i \in \bar{\tau}$ if and only if $(a_{ij}, a_{ji}) \in \{(-2, -1), (-2, -2), (-4, -1), (0, 0)\}$ for all $j \neq i$). Let τ be a subset of $\bar{\tau}$ (τ can be any subset of $\bar{\tau}$, i.e., $0 \leq |\tau| \leq |\bar{\tau}|$). Then there exist an affine root system R of rank l and a base $\Pi = \{\alpha_0, \dots, \alpha_l\}$ of R having the following properties:

(2-i) $a_{ij} = (\alpha_i^\vee, \alpha_j)$.

(2-ii) $R \cap \mathbb{R}\alpha_i = \begin{cases} \{\alpha_i, -\alpha_i\} & i \notin \tau, \\ \{\alpha_i, 2\alpha_i, -\alpha_i, -2\alpha_i\} & i \in \tau. \end{cases}$

(3) Let (R, V) be an affine root system of rank l . Then, for any two bases of Π_1 and Π_2 of R , there exist $\varepsilon \in \{1, -1\}$ and $w \in W_\Pi$ such that $\varepsilon w(\Pi_2) = \Pi_1$.

(4) Let (R, V) be an affine root system of rank l . Let Π be a base of R . Then we have:

(4-1) There exists $\alpha_0 \in \Pi$ such that $\pi(\Pi \setminus \{\alpha_0\})$ is a base of $\pi(R)$.

(4-2) There exists a unique $\delta(\Pi) \in \mathbb{N}\Pi$ such that

(2.2)
$$\mathbb{Z}\delta(\Pi) = V^0 \cap \mathbb{Z}R.$$

(4-3) $\alpha_0 = \alpha_0(R, \Pi \setminus \{\alpha_0\}, \delta(\Pi))$ (see (1-1) above).

(4-4) $A(\Pi)$ is a generalized Cartan matrix of affine-type (see (1.10) for $A(\Pi)$).

(4-5) For $\mathbb{X} = \mathbb{Z}$ or \mathbb{R} ,

(2.3)
$$\mathbb{X}R = \bigoplus_{\alpha \in \Pi} \mathbb{X}\alpha = \left(\bigoplus_{\alpha \in \Pi \setminus \{\alpha_0\}} \mathbb{X}\alpha \right) \bigoplus \mathbb{X}\delta(\Pi).$$

Proof of this theorem shall be given in Section 8.

As an immediate consequence of Theorem 2.1 (1)-(2), we have:

Lemma 2.3. (cf. [11, Appendixes 1 and 2]) Let $l \in \{1, 2\}$. Let (R, V) be an affine root system of rank l . Let $\Pi' = \{\alpha_1, \dots, \alpha_l\}$ be a subset of R such that $\pi(\Pi')$ is a base of $\pi(R)$ and, for $1 \leq i \leq l$, $2\alpha_i \in R$ if there exists $\alpha'_i \in R$ with $2\alpha'_i \in R$ and $\pi(\alpha'_i) = \pi(\alpha_i)$ (cf. (1.5)). Let $\delta \in \mathbb{Z}R$ be such that

$\mathbb{Z}\delta = \mathbb{Z}R \cap V^0$ and let $\alpha_0 = \alpha_0(R, \Pi', \delta)$. (So, $\delta = \delta(\Pi)$, where $\Pi = \{\alpha_0\} \cup \Pi'$ (see Theorem 2.1 (1) (1-1) and (2.2))). Then we have:

(1) Assume $l = 1$. Then

$$(2.4) \quad R = \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha_1 + \mathbb{Z}\delta) \cup (2\varepsilon\alpha_1 + g_1\delta))$$

($\in \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\delta$ (cf. (2.3))), where

$$(2.5) \quad g_1 \in \{\emptyset, 2\mathbb{Z} + 1, \mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}\}.$$

(Any g_1 of (2.5) can occur.) Moreover, if $g_1 \in \{\emptyset, 2\mathbb{Z}, 4\mathbb{Z}\}$ (resp. $\{2\mathbb{Z} + 1, \mathbb{Z}\}$), then $\alpha_0 = -\alpha_1 + \delta$ (resp. $\alpha_0 = -2\alpha_1 + \delta$).

(2) Assume $l = 2$ and $(\alpha_1, \alpha_1) \leq (\alpha_2, \alpha_2)$. See (1.1) for $W_{\{\alpha_1, \alpha_2\}}$. Then

$$(2.6) \quad R = \bigcup_{w \in W_{\{\alpha_1, \alpha_2\}}} \left(\bigcup_{i=1}^2 ((w(\alpha_i) + \mathbb{Z}k_i\delta) \cup (w(2\alpha_i) + g_i k_i \delta)) \right)$$

($\in \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\delta$ (cf. (2.3))), where

$$(2.7) \quad k_1 = 1, \quad k_2 \in \{1, -(\alpha_1^\vee, \alpha_2)\}, \quad g_2 = \emptyset$$

and

$$(2.8) \quad g_1 \in \begin{cases} \{\emptyset\} & \text{if } (\alpha_1^\vee, \alpha_2) \neq -2, \\ \{\emptyset, \mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z} + 1\} & \text{if } (\alpha_1^\vee, \alpha_2) = -2 \text{ and } k_2 = 1, \\ \{\emptyset, 2\mathbb{Z}, 4\mathbb{Z}\} & \text{if } (\alpha_1^\vee, \alpha_2) = -2 \text{ and } k_2 = 2. \end{cases}$$

(Any k_2 of (2.7) and any g_1 of (2.8) can occur.) Moreover, for $i \in \{1, 2\}$,

$$R \cap (\mathbb{R}\alpha_i \oplus \mathbb{R}\delta) = \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha_i + \mathbb{Z}k_i\delta) \cup (2\varepsilon\alpha_i + g_i k_i \delta)).$$

Furthermore, letting $x = -(\alpha_1^\vee, \alpha_2) (= \frac{(\alpha_2, \alpha_2)}{(\alpha_1, \alpha_1)} \in \{1, 2, 3\})$, we have:

$$(2.9) \quad \alpha_0 = \begin{cases} s_{\alpha_1}(-\alpha_2 + \delta) = s_{\alpha_2}(-\alpha_1 + \delta) & \text{if } k_2 = 1, x = 1, g_1 = \emptyset, \\ s_{\alpha_1}(-\alpha_2 + \delta) & \text{if } k_2 = 1, x = 2, g_1 \in \{\emptyset, 2\mathbb{Z}\}, \\ s_{\alpha_2}(-2\alpha_1 + \delta) & \text{if } k_2 = 1, x = 2, g_1 \in \{\mathbb{Z}, 2\mathbb{Z} + 1\}, \\ s_{\alpha_2} s_{\alpha_1}(-\alpha_2 + \delta) & \text{if } k_2 = 1, x = 3, g_1 = \emptyset, \\ s_{\alpha_2}(-\alpha_1 + \delta) & \text{if } k_2 = 2, x = 2, g_1 \in \{\emptyset, 2\mathbb{Z}, 4\mathbb{Z}\}, \\ s_{\alpha_1} s_{\alpha_2}(-\alpha_1 + \delta) & \text{if } k_2 = 3, x = 3, g_1 = \emptyset. \end{cases}$$

(3) (Here we give details of (1)-(2) above.) Assume $l \in \{1, 2\}$. Then R is exactly one of the following. In the list below, we let $\tau = \{i | 2\alpha_i \in R, 0 \leq i \leq l\}$ and let $A = A(\Pi)$ be of (1.10).

(3-1) The case of $l = 1$.

$(A_1^{(1)})$ $\alpha_0 = \delta - \alpha_1$, $R = \cup_{\varepsilon \in \{\pm 1\}} (\varepsilon\alpha_1 + \mathbb{Z}\delta)$, A is $A_1^{(1)}$, $\tau = \emptyset$.

$(A_2^{(2)})$ $\alpha_0 = \delta - 2\alpha_1$, $R = \cup_{\varepsilon \in \{\pm 1\}} ((\varepsilon\alpha_1 + \mathbb{Z}\delta) \cup (2\varepsilon\alpha_1 + (2\mathbb{Z} + 1)\delta))$,
 A is $A_2^{(2)}$, $\tau = \emptyset$.

$(B^{(1)}(0, 1))$ $\alpha_0 = \delta - 2\alpha_1$ and $R = \cup_{\varepsilon \in \{\pm 1\}} ((\varepsilon\alpha_1 + \mathbb{Z}\delta) \cup (2\varepsilon\alpha_1 + \mathbb{Z}\delta))$, A is $A_2^{(2)}$, $\tau = \{1\}$.

$(C^{(2)}(2))$ $\alpha_0 = \delta - \alpha_1$, $R = \cup_{\varepsilon \in \{\pm 1\}} ((\varepsilon\alpha_1 + \mathbb{Z}\delta) \cup (2\varepsilon\alpha_1 + 2\mathbb{Z}\delta))$, A is $A_1^{(1)}$,
 $\tau = \{0, 1\}$.

$(A^{(4)}(0, 2))$ $\alpha_0 = \delta - \alpha_1$, $R = \cup_{\varepsilon \in \{\pm 1\}} ((\varepsilon\alpha_1 + \mathbb{Z}\delta) \cup (2\varepsilon\alpha_1 + 4\mathbb{Z}\delta))$
 A is $A_1^{(1)}$, $\tau = \{1\}$.

(3-2) The case of $l = 2$. We assume $(\alpha_1, \alpha_1) \leq (\alpha_2, \alpha_2)$.

$(A_2^{(1)})$ $\alpha_0 = \delta - (\alpha_1 + \alpha_2)$, $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} (w(\alpha_1) + \mathbb{Z}\delta)$, A is $A_2^{(1)}$, $\tau = \emptyset$,
 $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$.

$(C_2^{(1)})$ $\alpha_0 = \delta - (2\alpha_1 + \alpha_2)$, $R = \cup_{i=1}^2 \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} (w(\alpha_i) + \mathbb{Z}\delta)$, A is $C_2^{(1)}$,
 $\tau = \emptyset$, $2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$.

$(G_2^{(1)})$ $\alpha_0 = \delta - (3\alpha_1 + 2\alpha_2)$, $R = \cup_{i=1}^2 \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} (w(\alpha_i) + \mathbb{Z}\delta)$, A is $G_2^{(1)}$,
 $\tau = \emptyset$, $3(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$.

$(A_4^{(2)})$ $\alpha_0 = \delta - (2\alpha_1 + 2\alpha_2)$, $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((\cup_{i=1}^2 w(\alpha_i) + \mathbb{Z}\delta) \cup (w(2\alpha_1) + (2\mathbb{Z} + 1)\delta))$, A is $A_4^{(2)}$, $\tau = \emptyset$, $2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$.

$(D_3^{(2)})$ $\alpha_0 = \delta - (\alpha_1 + \alpha_2)$, $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((w(\alpha_1) + \mathbb{Z}\delta) \cup (w(\alpha_2) + 2\mathbb{Z}\delta))$,
 A is $D_3^{(2)}$, $\tau = \emptyset$, $2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$.

$(D_4^{(3)})$ $\alpha_0 = \delta - (2\alpha_1 + \alpha_2)$, $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((w(\alpha_1) + \mathbb{Z}\delta) \cup (w(\alpha_2) + 3\mathbb{Z}\delta))$,
 A is $D_4^{(3)}$, $\tau = \emptyset$, $3(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$.

$(B^{(1)}(0, 2))$ $\alpha_0 = \delta - (2\alpha_1 + 2\alpha_2)$, $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((\cup_{i=1}^2 w(\alpha_i) + \mathbb{Z}\delta) \cup (w(2\alpha_1) + \mathbb{Z}\delta))$, A is $A_4^{(2)}$, $\tau = \{1\}$, $2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$.

$(A^{(2)}(0, 3))$ $\alpha_0 = \delta - (2\alpha_1 + \alpha_2)$, $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((\cup_{i=1}^2 w(\alpha_i) + \mathbb{Z}\delta) \cup (w(2\alpha_1) + 2\mathbb{Z}\delta))$, A is $C_2^{(1)}$, $\tau = \{1\}$, $2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$.

$(C^{(2)}(3))$ $\alpha_0 = \delta - (\alpha_1 + \alpha_2)$, $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((w(\alpha_1) + \mathbb{Z}\delta) \cup (w(\alpha_2) + 2\mathbb{Z}\delta) \cup (w(2\alpha_1) + 2\mathbb{Z}\delta))$, A is $D_3^{(2)}$, $\tau = \{0, 1\}$, $2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$.

$(A^{(4)}(0, 4)) \alpha_0 = \delta - (\alpha_1 + \alpha_2)$, $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} (((w(\alpha_1) + \mathbb{Z}\delta) \cup (w(\alpha_2) + 2\mathbb{Z}\delta) \cup (w(2\alpha_1) + 4\mathbb{Z}\delta)))$, A is $D_3^{(2)}$, $\tau = \{1\}$, $2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$.

(The symbol in each parenthesis is the notation used in [10, Table 1-4] for the Dynkin diagram for the pair (A, τ) ; see also Section 9.)

Proof. (Strategy. Case-by-case checking with the use of Theorem 2.1 (1)-(2).) Using Theorem 2.1 (1)-(2), we see that (3) holds. By checking the R 's of (3) directly, we have (1) and (2). \square

Let R be an affine root system with a base Π . By (1.3), we have

$$(2.10) \quad R \cap \left(\bigcup_{\alpha \in \Pi} \mathbb{N}\alpha \right) = \Pi \cup (2\Pi \cap R).$$

2.2 Key facts

The following lemma is a key to this paper.

Lemma 2.4. *Let $l \in \mathbb{N}$. Let (R, V) be an affine root system of rank l with a base Π (cf. (1.6)). Let S be a non-empty subset of Π (i.e., $1 \leq |S| \leq |\Pi| (= l + 1)$). Let W_S be as in (1.1). Let $\delta = \delta(\Pi)$ be of (2.2).*

(1) *If $\beta \in R \cap \mathbb{R}S (= R \cap \mathbb{Z}S)$ (cf. (2.3)), then there exists $w \in W_S$ such that*

$$(2.11) \quad w(\beta) \in S \cup (2S \cap R)$$

(see also (2.10)). (Namely, $R \cap \mathbb{R}S = W_S \cdot (S \cup (2S \cap R))$)

(2) *If $\xi \in \mathbb{Z}S \setminus (R \cup \mathbb{Z}\delta)$, then there exists $w \in W_S$ such that*

$$(2.12) \quad w(\xi) \in (\mathbb{Z}S \setminus (\mathbb{Z}_+S \cup \mathbb{Z}_-S)) \cup ((\cup_{\alpha \in S} \mathbb{N}\alpha) \setminus (S \cup (2S \cap R))).$$

(3) $W_{R \cap \mathbb{R}S} = W_S$. (In particular, $W = W_\Pi$, where we recall $W = W_R$ from Definition 1.1.)

(4) *If $\xi \in \mathbb{Z}S \setminus \mathbb{Z}\delta$, then there exists $w \in W_S$ such that*

$$(2.13) \quad w(\xi) \in \left(\left(\bigcup_{\alpha \in S} \mathbb{N}\alpha \right) \bigcup (\mathbb{Z}S \setminus (\mathbb{Z}_+S \cup \mathbb{Z}_-S)) \right).$$

Proof of this lemma shall be given in Section 8.

We also need the following lemma.

Lemma 2.5. *Let $l \in \mathbb{N}$. Let R be an affine root system of rank l with a base Π . (Notice $W = W_\Pi$ (see Lemma 2.4 (3)).) Let $\alpha, \beta \in \Pi$ be such that $\alpha \neq \beta$. Then (I) and (II) below are equivalent:*

- (I) *There exists $w \in W$ such that $w(\alpha) = \beta$.*
- (II) *There exist $\gamma_1, \dots, \gamma_r \in \Pi$ (with $r \geq 2$) such that $\gamma_1 = \alpha$, $\gamma_r = \beta$ and $(\gamma_i^\vee, \gamma_{i+1}) = (\gamma_i, \gamma_{i+1}^\vee) = -1$ for all $1 \leq i \leq r - 1$.*

Proof of this lemma shall be given in Section 8.

3 Elliptic root systems of rank ≥ 2

3.1 Maps k and g and construction of $R(\Pi, k, g)$

In this section, we always assume (R, V) to be an elliptic root system of rank $l \geq 2$. Let $\Pi \cup \{a\}$ be a fundamental-set of R (see Definition 1.2). Let $(\Pi_{\text{sh}})'$ be the subset of Π formed by $\alpha \in \Pi$ satisfying the condition that either $(\alpha, \beta) = 0$ or $(\beta, \beta) = 2(\alpha, \alpha)$ holds for every $\beta \in \Pi \setminus \{\alpha\}$. Then $(\Pi_{\text{sh}})' \subset \Pi_{\text{sh}}$ (see (1.47) for Π_{sh}) and, more precisely, we have

$$(3.1) \quad (\Pi_{\text{sh}})' = \begin{cases} \Pi_{\text{sh}} & \text{if } l = 2 \text{ and } A(\Pi) \text{ is } C_2^{(1)}, A_4^{(2)} \text{ or } D_3^{(2)}, \\ \Pi_{\text{sh}} & \text{if } l \geq 3 \text{ and } A(\Pi) \text{ is } B_l^{(1)}, A_{2l}^{(2)} \text{ or } D_{l+1}^{(2)}, \\ \emptyset & \text{otherwise} \end{cases}$$

(see (1.15), (1.45), (1.47) and Section 9 for the notation).

Theorem 3.1. *Let R be an elliptic root system of rank $l \geq 2$. Let $\Pi \cup \{a\}$ be a fundamental-set of R (see Definition 1.2). Recall (3.1) and (1.14). Then we can define maps $k : \Pi \rightarrow \mathbb{N}$ and $g : \Pi \rightarrow \{\emptyset, 2\mathbb{Z} + 1, \mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}\}$ by*

$$(3.2) \quad R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a) = \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha + \mathbb{Z}k(\alpha)a) \cup (2\varepsilon\alpha + g(\alpha)k(\alpha)a))$$

($\alpha \in \Pi$). Moreover k and g have the following properties:

(i)

$$(3.3) \quad R = \bigcup_{w \in W_\Pi} \left(\bigcup_{\alpha \in \Pi} ((w(\alpha) + \mathbb{Z}k(\alpha)a) \cup (w(2\alpha) + g(\alpha)k(\alpha)a)) \right).$$

- (ii) $\min k(\Pi) = 1$.
- (iii) If $\alpha, \beta \in \Pi$ are such that $\alpha \neq \beta$, $(\alpha, \beta) \neq 0$ and $(\alpha, \alpha) \leq (\beta, \beta)$, then $k(\beta) = k(\alpha)$ or $k(\beta) = -(\alpha^\vee, \beta)k(\alpha)$. (Notice that (since $l \geq 2$) $(\beta^\vee, \alpha) = -1$ and $(\alpha^\vee, \beta) \in \{-1, -2, -3\}$. See also Remark 3.1 (1) below.)
- (iv) $g(\alpha) = \emptyset$ if $\alpha \notin (\Pi_{\text{sh}})'$.
- (v) $g(\alpha) \in \{\emptyset, \mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z}+1\}$ if $\alpha \in (\Pi_{\text{sh}})'$ and there exists $\beta \in \Pi \setminus \{\alpha\}$ such that $(\alpha, \beta) \neq 0$ and $k(\alpha) = k(\beta)$ (notice $(\alpha^\vee, \beta) = -2$ and $(\beta^\vee, \alpha) = -1$).
- (vi) $g(\alpha) \in \{\emptyset, 2\mathbb{Z}, 4\mathbb{Z}\}$ if $\alpha \in (\Pi_{\text{sh}})'$ and there exists $\beta \in \Pi \setminus \{\alpha\}$ such that $(\alpha, \beta) \neq 0$ and $2k(\alpha) = k(\beta)$ (notice $(\alpha^\vee, \beta) = -2$ and $(\beta^\vee, \alpha) = -1$).
- (vii) $k(\Pi) \subset \{1, 2, 3, 4\}$.

Remark 3.1. Before we give the proof of Theorem 3.1, we remark the following:

(1) Since $l \geq 2$, it follows that $(\beta^\vee, \alpha) = -1$ and $(\alpha^\vee, \beta) \in \{-1, -2, -3\}$ for $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$, $(\alpha, \beta) \neq 0$ and $(\alpha, \alpha) \leq (\beta, \beta)$. The image $k(\Pi)$ for each Π is explicitly indicated in (3.13) in the proof of Theorem 3.1. In the cases where $g(\alpha) \neq \emptyset$ for some $\alpha \in \Pi$, all the possible values of k and g are given in Tables 4.1-4.4.

(2) By (iii) and (iv) of Theorem 3.1, we conclude that if $\alpha \in \Pi$ is such that $g(\alpha) \neq \emptyset$, then $(\beta^\vee, \alpha) = -1$, $(\alpha^\vee, \beta) = -2$ and $k(\beta) \in \{k(\alpha), 2k(\alpha)\}$ for $\beta \in \Pi \setminus \{\alpha\}$ with $(\alpha, \beta) \neq 0$.

(3) If $g(\alpha) \neq \emptyset$ for some $\alpha \in \Pi$, then $A(\Pi)$ is $C_2^{(1)}$ ($l = 2$), $B_l^{(1)}$ ($l \geq 3$), $A_{2l}^{(2)}$ ($l \geq 2$) or $D_{l+1}^{(2)}$ ($l \geq 2$) since $(\Pi_{\text{sh}})' \neq \emptyset$ (see (iv) of Theorem 3.1).

Proof of Theorem 3.1. Recall π_a from (1.11). (*Strategy.* This theorem follows mainly from Lemma 2.3 (1)-(2). Let $\alpha \in \Pi$ and $R^{\{\alpha\}} = R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a)$. We show that for each $\alpha \in \Pi$, $R^{\{\alpha\}}$ is a rank-one affine root system and $\pi_a(R^{\{\alpha\}})$ is a rank-one irreducible finite root system with the base $\{\pi_a(\alpha)\}$. Then, by Lemma 2.3 (1), we can define the maps k and g by (3.2). Let $\alpha, \beta \in \Pi$ be such that $\alpha \neq \beta$ and $(\alpha, \beta) \neq 0$. Set $R^{\{\alpha, \beta\}} := R \cap (\mathbb{R}\alpha \oplus \mathbb{R}\beta \oplus \mathbb{R}a)$. Then $R^{\{\alpha, \beta\}}$ is a rank-two affine root system and $\pi_a(R^{\{\alpha, \beta\}})$ is a rank-two irreducible finite root system with the base $\{\pi_a(\alpha), \pi_a(\beta)\}$. Then, by Lemma 2.3 (2), $R^{\{\alpha, \beta\}}$ has the same expression as (3.3) with $\{\alpha, \beta\}$ in place of Π . Then (iii)-(vi) follow from (2.7)-(2.8). Notice that for any $\bar{\mu} \in \pi_a(R) \setminus 2\pi_a(R)$, there exist $\bar{\alpha} \in \pi_a(\Pi)$ and $\bar{w} \in W_{\pi_a(\Pi)} (\subset \text{GL}(V/\mathbb{R}a))$ such that $\bar{\mu} = \bar{w}(\bar{\alpha})$ (see Lemma 2.4 (1)). Then (3.3) follows from (3.2). We can easily see that (ii) and (vii) hold.)

We proceed in steps.

Step 1. $\pi_a(R)$ is an affine root system with the base $\pi_a(\Pi)$. This is clear from (FS2) (see Definition 1.2).

Step 2. Let $\alpha \in \Pi$. If there exists $\alpha' \in R$ with $\pi_a(\alpha') = \pi_a(\alpha)$ and $2\alpha' \in R$, then $2\alpha \in R$. This is clear from (FS3).

Step 3. For each $\alpha \in \Pi$,

$$(3.4) \quad (\alpha + (\mathbb{Z} \setminus \{0\})a) \cap R \neq \emptyset.$$

By (1.14), R cannot be included in $\mathbb{Z}\Pi$. Hence there exist $\mu \in R$ and $m \in \mathbb{Z} \setminus \{0\}$ such that $\mu \in ma + \mathbb{Z}\Pi$. By Step 1 and Lemma 2.4 (1), we see that there exist $\gamma \in \Pi$, $c \in \{1, 2\}$ and $w \in W_\Pi$ such that $w(\mu) = c\gamma + ma$. Notice that

$$(3.5) \quad R \ni s_\gamma s_{c\gamma+ma}(\gamma) = s_\gamma(\gamma - (c^{-1}2)(c\gamma + ma)) = \gamma - 2c^{-1}ma.$$

(Hence (3.4) holds for this special γ .) Let $\lambda = \gamma - 2c^{-1}ma$. Notice that for $\beta \in \Pi$,

$$(3.6) \quad R \ni s_\gamma s_\lambda(\beta) = s_\gamma(\beta - (\gamma^\vee, \beta)\lambda) = \beta + (\gamma^\vee, \beta) \cdot 2c^{-1}ma.$$

Notice that for any $\alpha \in \Pi$, there exist $\gamma_1, \dots, \gamma_r \in \Pi$ such that $\gamma_1 = \alpha$, $\gamma_r = \gamma$ and $(\gamma_i, \gamma_{i+1}) \neq 0$ ($1 \leq i \leq r-1$). Then by (3.5) and (3.6), we see that (3.4) hold for all $\alpha \in \Pi$.

Step 4. $R^{\{\alpha, \beta\}}$ is a rank-two affine root system with $(\mathbb{R}R^{\{\alpha, \beta\}})^0 = \mathbb{R}a$, where $R^{\{\alpha, \beta\}}$ is as in Strategy. Let $\alpha, \beta \in \Pi$ be such that $\alpha \neq \beta$ and $(\alpha, \beta) \neq 0$. Since $l \geq 2$, we have $(\alpha^\vee, \beta)(\alpha^\vee, \beta) \in \{1, 2, 3\}$ (cf. (1.45) and Section 9), so

$$(3.7) \quad \det \begin{pmatrix} (\alpha^\vee, \alpha) & (\alpha^\vee, \beta) \\ (\beta^\vee, \alpha) & (\beta^\vee, \beta) \end{pmatrix} = 4 - (\alpha^\vee, \beta)(\alpha^\vee, \beta) \neq 0.$$

Let $V^{\{\alpha, \beta\}} = \mathbb{R}\alpha \oplus \mathbb{R}\beta \oplus \mathbb{R}a$ and $R^{\{\alpha, \beta\}} = R \cap V^{\{\alpha, \beta\}}$. By (3.4) and by noticing that $\alpha \in R$ also holds (cf. (FS2)), we see that $\mathbb{Z}R^{\{\alpha, \beta\}} \cap \mathbb{Z}a \neq \{0\}$. Hence there exists $k_{\alpha, \beta} \in \mathbb{N}$ such that

$$(3.8) \quad \mathbb{Z}R^{\{\alpha, \beta\}} = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}k_{\alpha, \beta}a$$

(see also (1.14)). By (3.7)-(3.8), since R satisfies (AX3,4), we see that $(R^{\{\alpha, \beta\}}, V^{\{\alpha, \beta\}})$ is a rank-two affine root system with $(V^{\{\alpha, \beta\}})^0 = \mathbb{R}a$, as desired.

Step 5. $\pi_a(R^{\{\alpha,\beta\}})$ is a rank-two irreducible finite root system with the base $\{\pi_a(\alpha), \pi_a(\beta)\}$, where $R^{\{\alpha,\beta\}}$ is as in Strategy. Let $R^{\{\alpha,\beta\}}$ be as in Step 4. By Step 4, $\pi_a(R^{\{\alpha,\beta\}})$ is a rank-two irreducible finite root system. By (1.14),

$$\pi_a(R^{\{\alpha,\beta\}}) \subset (\mathbb{Z}_+\pi_a(\alpha) \oplus \mathbb{Z}_+\pi_a(\beta)) \cup (\mathbb{Z}_-\pi_a(\alpha) \oplus \mathbb{Z}_-\pi_a(\beta)).$$

Hence $\{\pi_a(\alpha), \pi_a(\beta)\}$ is a base of $\pi_a(R^{\{\alpha,\beta\}})$ (cf. (1.6)).

Step 6. We can define the maps k and g by (3.2). Let $\alpha \in \Pi$ and $R^{\{\alpha\}} = R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a)$ (cf. (1.14)). By the same arguments as in Steps 4 and 5, we see the following:

(Step 6-1) $R^{\{\alpha\}}$ is a rank-one affine root system.

(Step 6-2) There exists $k(\alpha) \in \mathbb{N}$ such that

$$(3.9) \quad \mathbb{Z}R^{\{\alpha\}} = \mathbb{Z}\alpha \oplus \mathbb{Z}k(\alpha)a.$$

(Step 6-3) $\pi_a(R^{\{\alpha\}})$ is a rank-one irreducible finite root system with the base $\{\pi_a(\alpha)\}$.

Then, by Lemma 2.3 (1) and Step 2, there exists $g(\alpha) \in \{\emptyset, 2\mathbb{Z}+1, \mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}\}$ such that

$$(3.10) \quad R^{\{\alpha\}} = \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha + \mathbb{Z}k(\alpha)a) \cup (2\varepsilon\alpha + g(\alpha)k(\alpha)a)).$$

Thus we can define k and g by (3.2).

Step 7. (i) holds. By Step 1 and Lemma 2.4 (1), we see that for any $\bar{\mu} \in \pi_a(R) \setminus 2\pi_a(R)$, there exist $\alpha \in \Pi$ and $\bar{w} \in W_{\pi_a(\Pi)} (\subset \text{GL}(V/\mathbb{R}a))$ such that $\bar{w}(\bar{\mu}) = \bar{w}(\pi_a(\alpha))$. Then for any $\mu \in R$, there exist $\alpha \in \Pi$ and $w \in W_\Pi$ such that $w(\mu) \in R^{\{\alpha\}}$ (see Step 6 for $R^{\{\alpha\}}$). Then by (3.10), we have (3.3). Hence (i) holds.

Step 8. (iii)-(vi) hold. Let $R^{\{\alpha,\beta\}}$ be as in Steps 4 and 5. Assume $(\alpha, \alpha) \leq (\beta, \beta)$. By (1.14), (3.2), (3.8) and Steps 2, 4, 5 and by Lemma 2.3 (2), we have the following:

(Step 8-1) $k_{\alpha,\beta} = k(\alpha)$.

(Step 8-2)

$$(3.11) \quad R^{\{\alpha,\beta\}} = \bigcup_{w \in W_{\{\alpha,\beta\}}} \left(\bigcup_{\gamma \in \{\alpha,\beta\}} (w(\gamma) + \mathbb{Z}k(\gamma)a) \cup (w(2\gamma) + g(\gamma)k(\gamma)a) \right).$$

(Step 8-3)

$$(3.12) \quad \frac{k(\beta)}{k(\alpha)} \in \{1, -(\alpha^\vee, \beta)\}.$$

(Step 8-4) $g(\beta) = \emptyset$.

(Step 8-5)

$$g(\alpha) \in \begin{cases} \{\emptyset\} & \text{if } (\alpha^\vee, \beta) \neq -2, \\ \{\emptyset, \mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z} + 1\} & \text{if } (\alpha^\vee, \beta) = -2 \text{ and } \frac{k(\beta)}{k(\alpha)} = 1, \\ \{\emptyset, 2\mathbb{Z}, 4\mathbb{Z}\} & \text{if } (\alpha^\vee, \beta) = -2 \text{ and } \frac{k(\beta)}{k(\alpha)} = 2. \end{cases}$$

Then we see that k and g satisfy (iii)-(vi), as desired.

Step 9. (ii) *holds.* Let $\alpha \in \Pi$ be such that $k(\alpha) = \min k(\Pi)$. Using (iii), by case-by-case checking (cf. (1.45) and Section 9), we see that $k(\Pi) \subset \mathbb{N}k(\alpha)$, so $\mathbb{Z}k(\Pi) = \mathbb{Z}k(\alpha)$. On the other hand, we have $\mathbb{Z}R \cap \mathbb{Z}a = \mathbb{Z}a$ by (1.14) and we have $\mathbb{Z}R \cap \mathbb{Z}a = (\mathbb{Z}k(\Pi))a$ by (3.3). Hence $k(\alpha) = 1$, which implies (ii), as desired.

Step 10. (vii) *holds.* By (ii)-(iii) and case-by-case checking, we have

$$(3.13) \quad k(\Pi) = \begin{cases} \{1\} & \text{if } A(\Pi) \text{ is } A_l^{(1)}, D_l^{(1)} \text{ or } E_l^{(1)}, \\ \{1\} \text{ or } \{1, 3\} & \text{if } A(\Pi) \text{ is } G_2^{(1)} \text{ or } D_4^{(3)}, \\ \{1\}, \{1, 2\} \text{ or } \{1, 2, 4\} & \text{if } A(\Pi) \text{ is } A_{2l}^{(2)}, \\ \{1\} \text{ or } \{1, 2\} & \text{otherwise} \end{cases}$$

(see (1.45) (and Section 9) for the notation). Hence $k(\Pi) \subset \{1, 2, 3, 4\}$, which implies (vii), as desired. This completes the proof. \square

3.2 T_l, T_l^{norm} and $R(\Pi, k, g)$ and $\mathfrak{g}(\Pi, k, g)$ for any $(\Pi, k, g) \in T_l$

Definition 3.1. (T_l and a set $R(\Pi, k, g)$) (1) Recall the notation in Subsection 1.1. Let $l \geq 2$. Let V be a (fixed) $l + 2$ -dimensional \mathbb{R} -linear space equipped with a (fixed) positive semi-definite symmetric bilinear form (\cdot, \cdot) on V such that $\dim V^0 = 2$. Let a be a (fixed) element of $V^0 \setminus \{0\}$. Then:

$$(3.14) \quad \text{we define } T_l \text{ to be the set of the data } (\Pi, k, g)\text{'s}$$

such that each $(\Pi, k, g) \in T_l$ is formed by a subset $\Pi = \{\alpha_0, \dots, \alpha_l\}$ of V^\times (with $|\Pi| = l + 1$) and two maps $k : \Pi \rightarrow \{1, 2, 3, 4\}$ and $g : \Pi \rightarrow$

$\{\emptyset, 2\mathbb{Z} + 1, \mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}\}$ satisfying the following three conditions:

(1-i) $\Pi \cup \{a\}$ is an (\mathbb{R} -)basis of V .

(1-ii) $(a, \alpha_i) = 0$, $(\alpha_i, \alpha_i) > 0$ and $A(\Pi)$ is an $(l+1) \times (l+1)$ -generalized Cartan matrix of affine-type, where we set $A(\Pi) := ((\alpha_i^\vee, \alpha_j))_{0 \leq i, j \leq l}$ (see also (1.15)).

(1-iii) k and g have the same properties as in (i)-(vi) of Theorem 3.1.

(2) Let $l \geq 2$ and $(\Pi, k, g) \in T_l$ (see (3.14)). We denote by $R(\Pi, k, g)$ the subset of V defined by

$$(3.15) \quad R(\Pi, k, g) := \bigcup_{w \in W_\Pi} \left(\bigcup_{\alpha \in \Pi} ((w(\alpha) + \mathbb{Z}k(\alpha)a) \cup (w(2\alpha) + g(\alpha)k(\alpha)a)) \right)$$

(compare this with (1.19) and see Theorem 3.2 below). Let $\pi_a : V \rightarrow V/\mathbb{R}a$ be the canonical map (compare this with (1.11)). By Theorem 2.1 (2) and Lemma 2.4 (1), we see that

$$(3.16) \quad \pi_a(R(\Pi, k, g)) \text{ is an affine root system with the base } \pi_a(\Pi).$$

In particular,

$$(3.17) \quad R(\Pi, k, g) \subset V^\times.$$

By (3.16) and Theorem 2.1 (4), we can define $\delta(\Pi)$ and α_0 as follows: Let $\delta(\Pi) \in V^0$ be such that

$$(3.18) \quad \delta(\Pi) \in \mathbb{N}\Pi \text{ and } \mathbb{Z}\delta(\Pi) = V^0 \cap \mathbb{Z}\Pi$$

(compare this with (1.13)). In this paper, we always let $\alpha_0 \in \Pi$ be such that $\pi(\pi_a(\Pi \setminus \{\alpha_0\}))$ is a base of the irreducible finite root system $\pi(\pi_a(R))$ (see (1.5) for π), so

$$(3.19) \quad \pi_a(\alpha_0) = \alpha_0(\pi_a(R), \pi_a(\Pi \setminus \{\alpha_0\}), \pi_a(\delta(\Pi)))$$

(compare this with (1.12) and see Theorem 2.1 (4) (4-3)). For $\mu \in R(\Pi, k, g)$, let

$$(3.20) \quad p(\mu) = \begin{cases} 0 & \text{if } 2\mu \notin R(\Pi, k, g), \\ 1 & \text{if } 2\mu \in R(\Pi, k, g) \end{cases}$$

(compare this with (1.2) and see also Lemma 3.1 (2) below).

By (3.16) (with Theorem 2.1 (4) and (1.6)) and (3.18)-(3.19), we see that

$$(3.21) \quad \begin{cases} \text{the same equations as in (1.14) with} \\ R = R(\Pi, k, g) \text{ of (3.15) and } \delta = \delta(\Pi) \text{ of (3.18) hold.} \end{cases}$$

The following is a converse of Theorem 3.1.

Theorem 3.2. *Let $l \geq 2$ and $(\Pi, k, g) \in T_l$ (see (3.14)). Let $R(\Pi, k, g)$ be of (3.15). Then $(R(\Pi, k, g), V)$ is an elliptic root system and $\Pi \cup \{a\}$ is a fundamental-set of $R(\Pi, k, g)$ (see Definition 1.2).*

The proof of the theorem shall be given in Subsection 4.9.

Lemma 3.1. *Let $l \geq 2$ and $(\Pi, k, g) \in T_l$ (see (3.14)). Let $R = R(\Pi, k, g)$ be of (3.15). Let $p(\cdot)$ be of (3.20). See (1.1). Then we have:*

(1) For each $\alpha \in \Pi$,

$$(3.22) \quad R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a) = \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha + \mathbb{Z}k(\alpha)a) \cup (2\varepsilon\alpha + g(\alpha)k(\alpha)a)).$$

(2) For each $\alpha \in \Pi$ and $w \in W_\Pi$,

$$\begin{aligned} R \cap (\mathbb{R}w(\alpha) \oplus \mathbb{R}a) &= w(R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a)) \\ &= \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon w(\alpha) + \mathbb{Z}k(\alpha)a) \cup (\varepsilon w(2\alpha) + g(\alpha)k(\alpha)a)). \end{aligned}$$

In particular, if $\mu \in R$ is such that $p(\mu) = 1$ (resp. $p(\mu) = 0$), then $R \cap \mathbb{R}\mu$ is $\{-2\mu, -\mu, \mu, 2\mu\}$ (resp. $\{-\mu, \mu\}$ or $\{-\mu, -\frac{1}{2}\mu, \frac{1}{2}\mu, \mu\}$). (Compare this with (1.3).)

(3) For a subset S of Π ,

$$(3.23) \quad R \cap (\mathbb{R}S \oplus \mathbb{R}a) = \bigcup_{w \in W_S} \left(\bigcup_{\alpha \in S} ((w(\alpha) + \mathbb{Z}k(\alpha)a) \cup (w(2\alpha) + g(\alpha)k(\alpha)a)) \right).$$

(4) If $\mu, \nu \in R$ and $w \in \text{GL}(V)$ be such that $w(R) = R$ and $w(\mu) = \nu$, then $p(\mu) = p(\nu)$. In particular, $p(-\mu) = p(\mu)$ for $\mu \in R$.

Proof. (Strategy. We mainly use Lemma 2.5.)

(1) Since k and g have the same properties as those in (i)-(vi) of Theorem 3.1, by Lemma 2.5, we see that if $\alpha, \beta \in \Pi$ be such that $\beta \in W_\Pi.\alpha$, then $k(\alpha) = k(\beta)$ and $g(\alpha) = g(\beta)$. Then (3.22) is clear from (3.15).

(2) This follows from (1) and (3.15).

(3) This follows from (1), (3.15) and Lemma 2.4 (1) (with (3.16)).

(4) This is clear. □

Definition 3.2. (A Lie superalgebra $\mathfrak{g}(\Pi, k, g)$ associated with $(\Pi, k, g) \in T_l$ and a subset $T_l^{\text{nr}} of T_l)$

(1) Let $l \geq 2$ and $(\Pi, k, g) \in T_l$ (see (3.14)). Then:

$$(3.24) \quad \begin{cases} \text{we use the notation defined for this } (\Pi, k, g) \text{ in} \\ \text{the same way as in (1.20)-(1.21) and (1.24)-(1.29).} \end{cases}$$

Let $p(\cdot)$ be of (3.20). By (3.21)-(3.22), we see that

$$(3.25) \quad \text{this } p(\cdot) \text{ also satisfies the same equations as (1.22)-(1.23).}$$

Then (for this $(\Pi, k, g) \in T_l$):

$$(3.26) \quad \begin{cases} \text{we define the } \mathbb{C}\text{-Lie superalgebra } \mathfrak{g} = \mathfrak{g}(\Pi, k, g) \\ \text{and the subspaces } \mathfrak{h} = \mathfrak{h}(\Pi, k, g) \text{ and } \mathfrak{g}_\sigma \ (\sigma \in \mathcal{E}) \\ \text{of } \mathfrak{g} \text{ in the same way as in Definition 1.3.} \end{cases}$$

(*Caution:* we have not given a proof of Theorem 3.2.) Notice

$$(3.27) \quad \mathfrak{g}_\sigma = (\mathfrak{g}_\sigma \cap \mathfrak{g}(0)) \oplus (\mathfrak{g}_\sigma \cap \mathfrak{g}(1))$$

(cf. (1.26)).

(2) Let $l \geq 2$ and $(\Pi, k, g) \in T_l$ (see (3.14)). Define subsets $T_l^{\text{rm}}, T_l^{\text{nr}}, T_l^{\text{r}}$ and T_l^{nr} of T_l by

$$(3.28) \quad \begin{cases} T_l^{\text{rm}} = \{(\Pi, k, g) \in T_l | g(\alpha) = \emptyset \text{ for all } \alpha \in \Pi\}, \\ T_l^{\text{nr}} = T_l \setminus T_l^{\text{rm}}, \\ T_l^{\text{r}} = \{(\Pi, k, g) \in T_l | g(\alpha) \in \{\emptyset, 2\mathbb{Z} + 1\} \text{ for all } \alpha \in \Pi\}, \\ T_l^{\text{nr}} = T_l \setminus T_l^{\text{r}}. \end{cases}$$

In this paper, we mainly treat T_l^{nr} . Using (3.1), we can see:

$$(3.29) \quad \begin{cases} \text{If } (\Pi, k, g) \in T_l^{\text{nr}} \text{ and } l \geq 3 \text{ (resp. } l = 2), \\ \text{then } A(\Pi) \text{ is } B_l^{(1)}, A_{2l}^{(2)} \text{ or } D_{l+1}^{(2)} \text{ (resp. } C_2^{(1)}, A_4^{(2)} \text{ or } D_3^{(2)}) \end{cases}$$

(see (1.45), Section 9 and Definition 3.1 (1) (1-ii) for the notation).

Remark 3.2. (1) Let $l \geq 2$. Let J_l be the set of the isomorphism classes of the rank- l elliptic root systems. For a rank- l elliptic root system R , we denote by $[R]$ ($\in J_l$) the isomorphism class represented by R . By Theorem 3.2, we

can define a map $\mathcal{F}_l : T_l \rightarrow J_l$ by $\mathcal{F}_l((\Pi, k, g)) = [R(\Pi, k, g)]$. By Theorem 3.1, \mathcal{F}_l is a surjective map.

(2) Let $J_l^{\text{rm}} = \mathcal{F}_l(T_l^{\text{rm}})$, $J_l^r = \mathcal{F}_l(T_l^r)$ and $J_l^{\text{nr}} = \mathcal{F}_l(T_l^{\text{nr}})$. We see that J_l^r (respectively, J_l^{nr}) is the set of the isomorphism classes of the reduced (respectively, non-reduced) elliptic root systems of rank $l \geq 2$. We also see that J_l^{rm} is the set of the isomorphism classes $[R]$ of the reduced elliptic root systems (R, V) satisfying the condition that there exists a one-dimensional subspace G of V^0 such that $G \cap \mathbb{Z}R \neq \{0\}$ and $(\pi_G(R), V/G)$ is a reduced affine root system, where $\pi_G : V \rightarrow V/G$ is the canonical map. Notice that this assumption is the same as in [13, §5 A)]. We call this R a *reduced-marked* elliptic root system. Notice that $J_l^{\text{rm}} \subset J_l^r$ and, as mentioned in Remark 1.2 (2), $|J_l^r/J_l^{\text{rm}}| = 2$.

4 Root space decomposition (Proof of Theorem 1.1 (1))

4.1 Dimension of $\mathfrak{h} = \mathfrak{h}(\Pi, k, g)$

In Subsections 4.1-4.8, we always assume:

$$(4.1) \quad l \geq 2 \text{ and } (\Pi, k, g) \in T_l^{\text{rm}} \quad (\text{see (3.28)-(3.29)})$$

and we use the notation:

$$(4.2) \quad \left\{ \begin{array}{l} \text{Let } R = R(\Pi, k, g) \text{ be of (3.15). Let } \delta = \delta(\Pi) \text{ be of (3.18).} \\ \text{Let } \mathfrak{g} = \mathfrak{g}(\Pi, k, g) \text{ be of (3.26) (see also (4.3) below).} \\ \text{Let } \mathfrak{h} = \mathfrak{h}(\Pi, k, g) \text{ and } \mathfrak{g}_\sigma \ (\sigma \in \mathcal{E}) \text{ be of (3.26).} \\ \text{Let } \mathfrak{g}(i) \ (i \in \mathbb{Z}) \text{ be of (1.26) and notice } \mathfrak{g}(i+2) = \mathfrak{g}(i). \\ \text{Let } p(\mu) \ (\mu \in R) \text{ be of (3.20) (see also (3.25)).} \\ \text{Let } \mathcal{A}, \mathcal{E} = \mathcal{E}(\Pi, k, g), \mathcal{B}_+, c(\alpha) (\in \{1, 2\}) \\ \text{and } \alpha^* (= c(\alpha)\alpha + k(\alpha)a) \ (\alpha \in \Pi) \text{ be of (3.24).} \end{array} \right.$$

We begin with the following lemma.

Lemma 4.1. *See (4.1)-(4.2). We have $h_\sigma \neq 0$ for all $\sigma \in \mathcal{E} \setminus \{0\}$. In particular, $\dim \mathfrak{h} = l + 4$.*

Proof of this lemma shall be given in Subsection 5.9 (see Subsection 5.1 for strategy of the proof).

It is clear from (3.26) that

$$(4.3) \quad \mathfrak{g}(\Pi, k, g) = \bigoplus_{\lambda \in \mathbb{Z}\Pi \oplus \mathbb{Z}\mathfrak{a}} \mathfrak{g}_\lambda$$

and $\mathfrak{h} \subset \mathfrak{g}_0$ (we shall see $\mathfrak{h} = \mathfrak{g}_0$ in (4.69)).

4.2 Locally ad-nilpotent elements

Let \mathfrak{a} be a Lie superalgebra. Let $X \in \mathfrak{a}(i)$ for some $i \in \{0, 1\}$ (see (1.26) for $\mathfrak{a}(i)$). If for any $Y \in \mathfrak{a}$, there exists $m \in \mathbb{N}$ with $(\text{ad}X)^m(Y) = 0$, then we say that X is *locally ad-nilpotent*. If $X \in \mathfrak{a}(0)$ and X is locally ad-nilpotent, we can define $\exp(\text{ad}X) \in \text{Aut}(\mathfrak{a})$ by

$$(4.4) \quad \exp(\text{ad}X)(Y) = \sum_{i=0}^{\infty} \frac{1}{i!} (\text{ad}X)^i(Y)$$

(cf. [1, CHAPTER 1 §1], [8, §3.8]).

4.3 Automorphism $n_\mu \in \text{Aut}(\mathfrak{g})$

See (4.1)-(4.2). Let $\mu \in \mathcal{B}_+$. By (3.26), we have $E_{\pm\mu} \in \mathfrak{g}(p(\mu))$. By (SR1-5), we see that

$$(4.5) \quad E_{\pm\mu} \text{ are locally ad-nilpotent elements of } \mathfrak{g}$$

(see Subsection 4.2 for the terminology). By (4.5) and the representation theory of \mathfrak{sl}_2 or $\mathfrak{osp}(2|1)$, for $\nu \in \mathcal{E}$ and $X \in \mathfrak{g}_{\pm\nu} \setminus \{0\}$ with $[E_{\pm\mu}, X] = 0$, we can see

$$(4.6) \quad (\mu^\vee, \nu) \in (1 + p(\mu))\mathbb{Z}_+ \text{ and } (\text{ad}E_{\mp\mu})^{(\mu^\vee, \nu)+1}(X) = 0.$$

Set

$$E_{\pm\mu}^b := \begin{cases} E_{\pm\mu} & \text{if } p(\mu) = 0, \\ \pm \frac{1}{4} [E_{\pm\mu}, E_{\pm\mu}] & \text{if } p(\mu) = 1. \end{cases}$$

Then $E_{\pm\mu}^b (\in \mathfrak{g}(0))$ are locally ad-nilpotent (see (4.5)) (notice that if $p(\mu) = 1$, then $\text{ad}[E_{\pm\mu}, E_{\pm\mu}] = 2(\text{ad}E_{\pm\mu})^2$). We have

$$(4.7) \quad [E_\mu^b, E_{-\mu}^b] = h_{((1+p(\mu))\mu)^\vee} \quad \text{and} \quad [h_{((1+p(\mu))\mu)^\vee}, E_{\pm\mu}^b] = \pm 2E_{\pm\mu}^b.$$

By Lemma 4.1, $h_{((1+p(\mu))\mu)^\vee} \neq 0$. Hence, by (4.7),

$$(4.8) \quad E_{\pm\mu}^b \neq 0.$$

We also see that $(E_\mu^b, h_{(1+p(\mu))\mu^\vee}, E_{-\mu}^b)$ is an \mathfrak{sl}_2 -triple.

Let $\mu \in \mathcal{B}_+$. By the above argument, we can define $n_\mu \in \text{Aut}(\mathfrak{g})$ by

$$(4.9) \quad n_\mu = \exp(\text{ad}E_\mu^b) \exp(\text{ad}(-E_{-\mu}^b)) \exp(\text{ad}E_\mu^b)$$

(see also (4.4) and [1, CHAPTER 1 §1 (1.23)], [8, §3.8 Lemma 3.8 b])). We can see that

$$(4.10) \quad n_\mu^{\pm 1}(h_\sigma) = h_{s_\mu(\sigma)}, \quad \langle \mathfrak{g}_\sigma, E_\mu, E_{-\mu} \rangle \ni n_\mu^{\pm 1}(\mathfrak{g}_\sigma) = \mathfrak{g}_{s_\mu(\sigma)} \quad \text{and} \quad n_\mu^{\pm 1}(\mathfrak{g}(i)) = \mathfrak{g}(i)$$

for $\sigma \in \mathcal{E}$ and $i \in \{0, 1\}$ (cf. [1, CHAPTER 1 §1 (1.26) and PROPOSITION 1.27]), where $\langle \mathfrak{g}_\sigma, E_\mu, E_{-\mu} \rangle$ is the sub-Lie superalgebra of \mathfrak{g} generated by $E_\mu, E_{-\mu}$ and \mathfrak{g}_σ .

By (4.10), we have

$$(4.11) \quad \dim \mathfrak{g}_{w(\sigma)} = \dim \mathfrak{g}_\sigma \quad \text{and} \quad \dim(\mathfrak{g}_{w(\sigma)} \cap \mathfrak{g}(i)) = \dim(\mathfrak{g}_\sigma \cap \mathfrak{g}(i))$$

for $w \in W_{\mathcal{B}_+}$, $\sigma \in \mathcal{E}$ and $i \in \{0, 1\}$, where $W_{\mathcal{B}_+} = \langle s_\mu | \mu \in \mathcal{B}_+ \rangle$ (see (1.1)).

By the representation theory of \mathfrak{sl}_2 (cf. [18, Proof of Lemma 2.1]), for $\mu \in \mathcal{B}_+$, $\nu \in \mathcal{E}$ and $X \in \mathfrak{g}_{\pm\nu} \setminus \{0\}$ with $[E_{\pm\mu}^b, X] = 0$, letting $\bar{\mu} = (1+p(\mu))\mu$, we have $(\bar{\mu}^\vee, \nu) \in \mathbb{Z}_+$, $(\text{ad}E_{\mp\mu}^b)^{1+(\bar{\mu}^\vee, \nu)}(X) = 0$ and

$$(4.12) \quad n_\mu((\text{ad}E_{\mp\mu}^b)^i(X)) = (\mp 1)^{(\bar{\mu}^\vee, \nu)} (-1)^i \frac{i!}{((\bar{\mu}^\vee, \nu) - i)!} (\text{ad}E_{\mp\mu}^b)^{(\bar{\mu}^\vee, \nu) - i}(X)$$

($0 \leq i \leq (\bar{\mu}^\vee, \nu)$). For $\mu, \nu \in \mathcal{B}_+$ with $(\mu^\vee, \nu) = -1$ (so $(\nu^\vee, \mu) = \{-1, -2\}$), by (4.12), since $[E_{\pm\mu}, E_{\mp\nu}] = 0$ (by (SR5)) and $p(\mu) = 0$ (by (3.25) or (4.6)), we have

$$\begin{cases} n_\mu(E_{\pm\nu}) = \pm[E_{\pm\mu}, E_{\pm\nu}], \\ n_\nu(E_{\pm\mu}) = (\pm 1)^{(\nu^\vee, \mu)} \frac{1}{(-(\nu^\vee, \mu))!} (\text{ad}E_{\pm\nu})^{-(\nu^\vee, \mu)}(E_{\pm\mu}), \end{cases}$$

where we notice that $(\nu^\vee, \mu) = -2$ if $p(\nu) = 1$. Hence for $\mu, \nu \in \mathcal{B}_+$ with $(\mu^\vee, \nu) \leq 0$, we have

$$(4.13) \quad n_\nu(E_{\pm\mu}) = (\pm 1)^{(\nu^\vee, \mu)} \frac{1}{(-(\nu^\vee, \mu))!} (\text{ad}E_{\pm\nu})^{-(\nu^\vee, \mu)}(E_{\pm\mu}).$$

4.4 Significance of (SR5-9)

See (4.1)-(4.2). Let $(\alpha, \beta, y) \in \mathcal{A}$ (cf. (4.2), (3.24) and (1.24)). Then $(\alpha, \beta) \neq 0$, $\frac{(\beta, \beta)}{(\alpha, \alpha)} \in \{1, 2\}$ and $y = \frac{k(\beta)}{k(\alpha)}$. The following four cases exist:

$$(4.14) \quad \begin{cases} \text{(i)} & c(\alpha) = 1, (\alpha^\vee, \beta) = -1, y = 1 \text{ and } g(\alpha) = \emptyset, \\ \text{(ii)} & c(\alpha) = 1, (\alpha^\vee, \beta) = -2, y = 1 \text{ and } g(\alpha) \in \{\emptyset, 2\mathbb{Z}\}, \\ \text{(iii)} & c(\alpha) = 1, (\alpha^\vee, \beta) = -2, y = 2 \text{ and } g(\alpha) \in \{\emptyset, 2\mathbb{Z}, 4\mathbb{Z}\}, \\ \text{(iv)} & c(\alpha) = 2, (\alpha^\vee, \beta) = -2, y = 1 \text{ and } g(\alpha) \in \{\mathbb{Z}, 2\mathbb{Z} + 1\}, \end{cases}$$

where we have $c(\beta) = 1$, $(\beta^\vee, \alpha) = -1$ and $g(\beta) = \emptyset$ in any case of (i)-(iv). (*Proof.* Recall (1-iii) of Definition 3.1 (1) (see also (iii)-(vi) of Theorem 3.1) and notice that $c(\alpha) = 2$ if and only if $g(\alpha) \in \{\mathbb{Z}, 2\mathbb{Z} + 1\}$ (cf. (1.20)).) Let

$$(4.15) \quad \begin{cases} \alpha' = \beta \text{ and } \beta' = \alpha & \text{if (ii) of (4.14) holds,} \\ \alpha' = \alpha \text{ and } \beta' = \beta & \text{otherwise.} \end{cases}$$

By (4.14), we have $c(\beta') = 1$ and

$$(4.16) \quad \begin{cases} y = \frac{k(\beta')}{k(\alpha')} = -\frac{((\alpha')^\vee, \beta')}{c(\alpha')} = -(((\alpha')^*)^\vee, \beta') \text{ and} \\ -((\alpha')^\vee, (\beta')^*) = c(\alpha')y. \end{cases}$$

Hence we have

$$(4.17) \quad s_{(\alpha')^*}(\beta') = \beta' + y(\alpha')^* = (\beta')^* + c(\alpha')y\alpha' = s_{\alpha'}((\beta')^*).$$

On the other hand, by (4.13), (4.16) and (SR6,7), we have

$$(4.18) \quad n_{\alpha'}(E_{\pm(\beta')^*}) = n_{(\alpha')^*}(E_{\pm\beta'}).$$

This is a significance of (SR6,7).

Remark 4.1. We see that α' of (4.15) is λ of Remark 1.1 (2). In the case (i) of (4.14), we may change α and β in (SR6,7).

As a significance of (SR8,9), it follows that

$$(4.19) \quad \pm(i\alpha + (y - i)\alpha^* + \beta) \notin R \quad \text{if } 1 \leq i \leq y - 1.$$

(*Proof.* Since $y \geq 2$, (iii) of (4.14) holds. Hence, since $\frac{i}{y} \notin \mathbb{Z}$, by (3.2), we have $\pm(\beta - ik(\alpha)a) = \pm(\beta - \frac{i}{y}k(\beta)a) \notin R$. Hence $\pm(i\alpha + (y - i)\alpha^* + \beta) = \pm(\beta + y\alpha - ik(\alpha)a) = s_\alpha(\pm(\beta - ik(\alpha)a)) \notin R$ (cf. (AX4)).)

For a significance of (SR5), see (7.5).

4.5 Affine sub-Lie superalgebras

Lemma 4.2. *See (4.1)-(4.2). Let $r \in \mathbb{N}$. Let P be a subset of R formed by linearly independent $r + 1$ -elements such that the $(r + 1) \times (r + 1)$ -matrix $((\mu^\vee, \nu))_{\mu, \nu \in P}$ is a generalized Cartan matrix of affine-type (notice that it follows from (3.17) that $P \subset V^\times$). Let $\bar{E}_{\pm\mu}$ ($\mu \in P$) be elements of $\mathfrak{g}_{\pm\mu}$ satisfying the conditions (i)-(iii) below:*

- (i) *For each $\mu \in P$, there exists $\bar{p}(\mu) \in \{0, 1\}$ such that $\bar{E}_{\pm\mu} \in \mathfrak{g}(\bar{p}(\mu))$.*
- (ii) *$\bar{E}_{\pm\mu}$ are locally ad-nilpotent elements of \mathfrak{g} (see Subsection 4.2 for the terminology).*
- (iii) *The equations*

$$(4.20) \quad [\bar{E}_\mu, \bar{E}_{-\nu}] = \delta_{\mu, \nu} h_{\mu^\vee}.$$

hold.

Let $P' = \{\mu \in P \mid \bar{p}(\mu) = 1\}$. Let

$$(4.21) \quad R_P = W_P.(P \cup 2P')$$

($\subset \mathbb{Z}P$), where $W_P = \langle s_\mu \mid \mu \in P \rangle$ (see (1.1)). Let $\delta(P) \in \mathbb{N}P$ be such that $\mathbb{Z}\delta(P) = \mathbb{Z}P \cap V^0$ (cf. Lemma 2.2 (2)). Let \mathfrak{g}_P be the sub-Lie superalgebra of \mathfrak{g} generated by \mathfrak{h} and $\bar{E}_{\pm\mu}$ ($\mu \in P$). Then we have the following:

- (1) *($R_P, \mathbb{R}P$) is an affine root system.*
- (2) *P is a base of R_P (cf. (1.6)).*
- (3) *$R_P \cap 2P = 2P'$.*
- (4) *$\mathbb{Z}R_P \cap V^0 = \mathbb{Z}\delta(P)$.*
- (5)

$$(4.22) \quad \mathfrak{g}_P = \mathfrak{h} \bigoplus \left(\bigoplus_{\beta \in R_P} (\mathfrak{g}_P)_\beta \right) \bigoplus \left(\bigoplus_{m \in \mathbb{Z} \setminus \{0\}} (\mathfrak{g}_P)_{m\delta(P)} \right),$$

where $(\mathfrak{g}_P)_\sigma = \mathfrak{g}_P \cap \mathfrak{g}_\sigma$.

- (6) *$\dim(\mathfrak{g}_P)_\beta = 1$ for $\beta \in R_P$.*
- (7) *If $\beta \in R_P \setminus \frac{1}{2}R_P$ (resp. $\beta \in R_P \cap \frac{1}{2}R_P$), then $(\mathfrak{g}_P)_\beta \subset \mathfrak{g}(0)$ (resp. $(\mathfrak{g}_P)_\beta \subset \mathfrak{g}(1)$).*
- (8) *$(\mathfrak{g}_P)_{m\delta(P)} \subset \mathfrak{g}(m \sum_{\mu \in P} x_\mu \bar{p}(\mu))$ for $m \in \mathbb{Z}$, where $x_\mu \in \mathbb{N}$ are such that $\sum_{\mu \in P} x_\mu \mu = \delta(P)$.*

By (i), (ii), (iii) (i.e., (4.20)) and (4.6), we have

$$(4.23) \quad (\bar{E}_{\pm\mu})^{1-(\mu^\vee, \nu)} \bar{E}_{\pm\nu} = 0 \quad \text{for } \mu, \nu \in P \text{ with } \mu \neq \nu.$$

Then we can easily prove this lemma by using [10, Proposition 1.7] and Lemma 4.1. We also give a (self-contained) proof in Section 8, because the idea of the proof is a prototype of the one of Theorem 1.1 (1) (cf. Subsection 1.8).

4.6 Rank one and two subalgebras

Definition 4.1. See (4.1)-(4.2). Let S be a non-empty proper subset of Π .

(1) Let

$$(4.24) \quad R^S = R \cap \left(\left(\bigoplus_{\alpha \in S} \mathbb{Z}\alpha \right) \bigoplus \mathbb{Z}a \right) (= R \cap (\mathbb{R}S \bigoplus \mathbb{R}a))$$

(recall (3.21)).

(2) Let

$$(4.25) \quad \left\{ \begin{array}{l} \text{we define } \mathfrak{g}^S \text{ to be the sub-Lie superalgebra of } \mathfrak{g} \text{ generated by} \\ \mathfrak{h} \text{ and } E_\mu, E_{-\mu}, E_{\mu^*}, E_{-\mu^*} \text{ with all } \mu \in S. \end{array} \right.$$

For $\sigma \in \mathcal{E}$, let $\mathfrak{g}_\sigma^S = \mathfrak{g}_\sigma \cap \mathfrak{g}^S$.

Now we treat $R^{\{\alpha\}}$ and $\mathfrak{g}^{\{\alpha\}}$ with $\alpha \in \Pi$.

Lemma 4.3. See (1.1) and (4.1)-(4.2). Let $\alpha \in \Pi$.

(1) Let $R^{\{\alpha\}}$ be as in (4.24). Then $R^{\{\alpha\}} = (R^{\{\alpha\}}, \mathbb{R}\alpha \oplus \mathbb{R}a)$ is a rank-one affine root system with a base $\{\alpha, -\alpha^*\}$. Moreover, we have

$$(4.26) \quad R^{\{\alpha\}} = W_{\{\alpha, -\alpha^*\}} \cdot (\{\alpha, -\alpha^*\} \cup \{2\mu \mid \mu \in \{\alpha, -\alpha^*\}, p(\mu) = 1\}),$$

$$(4.27) \quad -\alpha^* = \alpha_0(R^{\{\alpha\}}, \{\alpha\}, -k(\alpha)a) \quad (\text{see Theorem 2.1 (1) (1-1) for RHS}),$$

$$(4.28) \quad R^{\{\alpha\}} = \cup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha + \mathbb{Z}k(\alpha)a) \cup (2\varepsilon\alpha + g(\alpha)k(\alpha)a))$$

and

$$(4.29) \quad \mathbb{Z}R^{\{\alpha\}} = \mathbb{Z}\alpha \oplus \mathbb{Z}k(\alpha)a, \quad \mathbb{Z}R^{\{\alpha\}} \cap V^0 = \mathbb{Z}k(\alpha)a.$$

(2) Let $\mathfrak{g}^{\{\alpha\}}$ be as in (4.25). Then we have

$$(4.30) \quad \mathfrak{g}^{\{\alpha\}} = \mathfrak{h} \bigoplus \left(\bigoplus_{\mu \in R^{\{\alpha\}}} \mathfrak{g}_{\mu}^{\{\alpha\}} \right) \bigoplus \left(\bigoplus_{m \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_{mk(\alpha)a}^{\{\alpha\}} \right)$$

and

$$(4.31) \quad \dim \mathfrak{g}_{\mu}^{\{\alpha\}} = 1 \text{ and } \mathfrak{g}_{\mu}^{\{\alpha\}} \subset \mathfrak{g}(p(\mu)) \text{ for } \mu \in R^{\{\alpha\}}.$$

Moreover $\mathfrak{g}_{mk(\alpha)a}^{\{\alpha\}} \subset \mathfrak{g}(m(p(\alpha^*) - c(\alpha)p(\alpha)))$ ($m \in \mathbb{Z}$).

Proof. (1) (*Strategy.* We mainly use Lemma 2.3 (1).)

The equality (4.28) follows from (3.22). Then (4.29) follows from (4.28) (and (3.21)). By (4.28) and Lemma 2.3 (1), we see that $R^{\{\alpha\}}$ is a rank-one affine root system with a base $\{\alpha, -\alpha^*\}$. Hence (4.26) follows from Lemma 2.4 (1) and, moreover, (4.27) follows from (4.29).

(2) (*Strategy.* We use Lemma 4.2 for $P = \{\alpha, -\alpha^*\}$.) It is clear that $((\mu^{\vee}, \nu))_{\mu, \nu \in \{\alpha, -\alpha^*\}}$ is a 2×2 -generalized Cartan matrix of affine-type. Then (2) follows from (1) of this lemma, (SR1-5), (4.5) and Lemma 4.2 for $P = \{\alpha, -\alpha^*\}$ and $\bar{E}_{\pm\alpha} = E_{\pm\alpha}$, $\bar{E}_{\pm\alpha^*} = (\pm 1)^{p(\alpha^*)} E_{\pm\alpha^*}$, where we notice $p(-\mu) = p(\mu)$ for $\mu \in P$ (see (3.25) or Lemma 3.1 (4)). \square

Lemma 4.4. See (1.1), (4.1)-(4.2) and (4.24). We have:

(1) $R = W_{\Pi} \cdot (\cup_{\alpha \in \Pi} R^{\{\alpha\}})$. Moreover, for $i \in \{0, 1\}$,

$$(4.32) \quad \{\mu \in R | p(\mu) = i\} = W_{\Pi} \cdot (\cup_{\alpha \in \Pi} \{\nu \in R^{\{\alpha\}} | p(\nu) = i\}).$$

(2) For $\lambda \in (\mathbb{Z}\Pi \oplus \mathbb{Z}a) \setminus (R \cup (\mathbb{Z}\delta \oplus \mathbb{Z}a))$, there exists $w \in W_{\Pi}$ such that

$$(4.33) \quad w(\lambda) \in \left(\bigcup_{\alpha \in \Pi} ((\mathbb{Z}\alpha \oplus \mathbb{Z}a) \setminus (R^{\{\alpha\}} \cup \mathbb{Z}a)) \right) \bigcup \left((\mathbb{Z}\Pi \oplus \mathbb{Z}a) \setminus ((\mathbb{Z}_+\Pi \oplus \mathbb{Z}a) \cup (\mathbb{Z}_-\Pi \oplus \mathbb{Z}a)) \right).$$

(3) $R = W_{\mathcal{B}_+} \cdot (\mathcal{B}_+ \cup \{2\mu | \mu \in \mathcal{B}_+, p(\mu) = 1\})$.

(4) $W_R = W_{\mathcal{B}_+}$ (notice $W = W_R$ (see Definition 1.1)).

Proof. (*Strategy.* This lemma follows easily from Lemma 2.4 (4).)

(1) This follows from (3.15), (4.28) and Lemma 3.1 (4).

(2) By (3.16) and by (2.13) for $\pi_a(R)$ and $S = \pi_a(\Pi)$, we see that for $\lambda \in (\mathbb{Z}\Pi \oplus \mathbb{Z}a) \setminus (\mathbb{Z}\delta \oplus \mathbb{Z}a)$, there exists $w \in W_\Pi$ such that

$$w(\lambda) \in \left(\bigcup_{\alpha \in \Pi} (\mathbb{N}\alpha \oplus \mathbb{Z}a) \right) \bigcup \left((\mathbb{Z}\Pi \oplus \mathbb{Z}a) \setminus ((\mathbb{Z}_+\Pi \oplus \mathbb{Z}a) \cup (\mathbb{Z}_-\Pi \oplus \mathbb{Z}a)) \right).$$

Hence (2) holds, where we recall $R^{\{\alpha\}} = R \cap (\mathbb{Z}\alpha \oplus \mathbb{Z}a)$ ($\alpha \in \Pi$) (see (4.24)).

(3) This follows from (1) and (4.26).

(4) This follows from (3) and the fact that $ws_\mu w^{-1} = s_{w(\mu)}$ ($w \in W_{\mathcal{B}_+}$, $\mu \in \mathcal{B}_+$). \square

In the following two lemmas, we treat $R^{\{\alpha, \beta\}}$ and $\mathfrak{g}^{\{\alpha, \beta\}}$.

Lemma 4.5. *See (1.1) and (4.1)-(4.2). Let $\alpha, \beta \in \Pi$ be such that $\alpha \neq \beta$, $(\alpha, \beta) \neq 0$ and $(\alpha, \alpha) \leq (\beta, \beta)$. Then we have the following;*

(1) *Let $R^{\{\alpha, \beta\}}$ be as in (4.24). Let $\alpha', \beta' \in \{\alpha, \beta\}$ be of (4.15). Let*

$$(4.34) \quad \gamma = s_{\beta'}(-(\alpha')^*).$$

Let $P = \{\alpha, \beta, \gamma\}$. Then $R^{\{\alpha, \beta\}} = (R^{\{\alpha, \beta\}}, \mathbb{R}\alpha \oplus \mathbb{R}\beta \oplus \mathbb{R}a)$ is a rank-two affine root system with the base P . Moreover, we have

$$(4.35) \quad R^{\{\alpha, \beta\}} = W_P \cdot (P \cup \{2\mu \mid \mu \in P, p(\mu) = 1\}),$$

$$(4.36) \quad R^{\{\alpha, \beta\}} = \bigcup_{w \in W_{\{\alpha, \beta\}}} \left(\bigcup_{\mu \in \{\alpha, \beta\}} ((w(\mu) + \mathbb{Z}k(\mu)a) \cup (2w(\mu)\alpha + g(\mu)k(\mu)a)) \right)$$

and

$$(4.37) \quad \mathbb{Z}R^{\{\alpha, \beta\}} = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}k(\alpha)a, \quad \mathbb{Z}R^{\{\alpha, \beta\}} \cap V^0 = \mathbb{Z}k(\alpha)a.$$

Furthermore, $\gamma = \alpha_0(R^{\{\alpha, \beta\}}, \{\alpha, \beta\}, -k(\alpha)a)$ (cf. Theorem 2.1 (1) (1-1)).

(2) *Let $\mathfrak{g}^{\{\alpha, \beta\}}$ be as in (4.25). Then we have*

$$(4.38) \quad \mathfrak{g}^{\{\alpha, \beta\}} = \mathfrak{h} \bigoplus \left(\bigoplus_{\mu \in R^{\{\alpha, \beta\}}} \mathfrak{g}_\mu^{\{\alpha, \beta\}} \right) \bigoplus \left(\bigoplus_{m \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_{mk(\alpha)a}^{\{\alpha, \beta\}} \right)$$

and

$$(4.39) \quad \dim \mathfrak{g}_\mu^{\{\alpha, \beta\}} = 1 \quad \text{for } \mu \in R^{\{\alpha, \beta\}}.$$

Moreover it follows that $\mathfrak{g}_\mu^{\{\alpha, \beta\}} \subset \mathfrak{g}(p(\mu))$ ($\mu \in R^{\{\alpha, \beta\}}$) and that for $m \in \mathbb{Z}$, there exists $j \in \{0, 1\}$ with $\mathfrak{g}_{mk(\alpha)a}^{\{\alpha, \beta\}} \subset \mathfrak{g}(j)$.

Proof. By (3.29), we have

$$(4.40) \quad \left\{ \frac{(\beta, \beta)}{(\alpha, \alpha)}, \frac{k(\beta)}{k(\alpha)} \right\} \subset \{1, 2\}.$$

(1) (*Strategy.* This follows easily from Lemma 2.3 (2).)

The equality (4.36) follows from (3.23). The equality (4.37) follows from (4.36) and (4.40). Comparing (4.36) with (2.6), by Lemma 2.3 (2), we see that $R^{\{\alpha, \beta\}}$ is a rank-two affine root system with the base P ; we also see $\gamma = \alpha_0(R^{\{\alpha, \beta\}}, \{\alpha, \beta\}, -k(\alpha)a)$. Then (4.35) follows from Lemma 2.4 (1).

(2) (*Strategy.* Let γ be of (4.34). Let $E_{\pm\gamma} = (\pm 1)^{p((\alpha')^*)} n_{\beta'}(E_{\mp(\alpha')^*})$. We use Lemma 4.2 for $P = \{\alpha, \beta, \gamma\}$ and $\bar{E}_{\pm\mu} = E_{\pm\mu}$ ($\mu \in P$); we can easily see that $\mathfrak{g}^{\{\alpha, \beta\}}$ is generated by \mathfrak{h} and $\{E_{\pm\mu} | \mu \in P\}$. To use Lemma 4.2, we show by case-by-case checking that $E_{\pm\mu}$ ($\mu \in P$) satisfy the equality (4.20).)

Let $P = \{\alpha, \beta, \gamma\}$. Let

$$(4.41) \quad E_{\pm\gamma} = (\pm 1)^{p((\alpha')^*)} n_{\beta'}(E_{\mp(\alpha')^*}).$$

We proceed in steps.

Step 1. $E_{\pm\gamma} \in \mathfrak{g}_{\pm\gamma} \cap \mathfrak{g}(p(\gamma))$. This follows from (4.10), (4.34) and (4.41), where we notice that $p(\mp(\alpha')^*) = p(\gamma)$ (see Lemma 3.1 (4)) and $E_{\mp(\alpha')^*} \in \mathfrak{g}(p((\alpha')^*))$ (see (3.26)).

Step 2. $\mathfrak{g}^{\{\alpha, \beta\}}$ is generated by \mathfrak{h} and $\{E_{\pm\mu} | \mu \in P\}$.

(*Strategy of Step 2.* The key in this step is (4.18).)

By (4.41) and (4.13), we have

$$(4.42) \quad E_{\pm\gamma} = (\pm 1)^{p((\alpha')^*)} (\mp 1)^x x^{-1} (\text{ad} E_{\mp\beta'})^x (E_{\mp(\alpha')^*}) \in \mathfrak{g}^{\{\alpha, \beta\}},$$

where $x = -((\beta')^\vee, (\alpha')^*) \in \{1, 2\}$. By (4.42), we can define $(\mathfrak{g}^{\{\alpha, \beta\}})'$ to be the sub-Lie superalgebra of $\mathfrak{g}^{\{\alpha, \beta\}}$ generated by \mathfrak{h} and $\{E_{\pm\mu} | \mu \in P\}$. By (4.41), we have

$$(4.43) \quad E_{\pm(\alpha')^*} = (\mp 1)^{p((\alpha')^*)} n_{\beta'}^{-1}(E_{\mp\gamma}) \in (\mathfrak{g}^{\{\alpha, \beta\}})'$$

By (4.18) and (4.43), we have

$$(4.44) \quad E_{\pm(\beta')^*} = n_{\alpha'}^{-1} n_{(\alpha')^*}(E_{\pm\beta'}) \in (\mathfrak{g}^{\{\alpha, \beta\}})'$$

By (4.43) and (4.44), the claim of this step holds, i.e., $\mathfrak{g}^{\{\alpha, \beta\}} = (\mathfrak{g}^{\{\alpha, \beta\}})'$.

Step 3. $E_{\pm\mu}$ ($\mu \in P$) satisfy the conditions (i) and (ii) of Lemma 4.2. By (3.26) and (4.5), for $\nu \in \{\alpha, \beta, \alpha^*, \beta^*\}$, we see that $E_{\pm\nu}$ satisfy those conditions. By (4.41) (with (4.5)) and Step 1, $E_{\pm\gamma}$ also satisfy those conditions.

Step 4. $E_{\pm\mu}$ ($\mu \in P$) satisfy the condition (iii) of Lemma 4.2 (i.e., (4.20)).

(*Strategy of Step 4.* We show (4.20) with $E_{\pm\mu}$ (in place of $\bar{E}_{\pm\mu}$) for $\mu \in P = \{\alpha, \beta, \gamma\}$ by case-by-case checking; we mainly use (SR6-9).)

By (SR1-5), we only need to show that

$$(4.45) \quad [E_\gamma, E_{-\gamma}] = h_{\gamma\nu},$$

$$(4.46) \quad [E_\gamma, E_{-\alpha}] = 0,$$

$$(4.47) \quad [E_\gamma, E_{-\beta}] = 0,$$

$$(4.48) \quad [E_{-\gamma}, E_\alpha] = 0,$$

and

$$(4.49) \quad [E_{-\gamma}, E_\beta] = 0.$$

The equality (4.45) follows from (SR4), (4.34), (4.41) and (4.10). In the following, we show (4.48) and (4.49); we can show (4.46) and (4.47) similarly. We let $X \sim Y$ mean that $X = xY$ for some $x \in \mathbb{C} \setminus \{0\}$. See Lemma 2.3 (3) (3-2) for the name of $R^{\{\alpha, \beta\}}$ (which is $A_2^{(1)}, \dots$, or $A^{(4)}(0, 4)$). We may identify $\gamma, \alpha, \beta, -k(\alpha)a, k(\alpha), g(\alpha), k(\beta)$ and $g(\beta)$ with $\alpha_0, \alpha_1, \alpha_2, \delta, k_1, g_1, k_2$ and g_2 in the statement of Lemma 2.3 (3) (3-2) respectively.

Case 1: $R^{\{\alpha, \beta\}}$ is $A_2^{(1)}$ (i.e., (i) of (4.14)). By (4.42) and (SR6), we have

$$(4.50) \quad E_{-\gamma} \sim [E_\beta, E_{\alpha^*}] = [E_{\beta^*}, E_\alpha].$$

Then (4.48) and (4.49) follows from (4.50) and (SR5).

Case 2: $R^{\{\alpha, \beta\}}$ is $C_2^{(1)}$ or $A^{(2)}(0, 3)$ (i.e., (ii) of (4.14)). We have

$$(4.51) \quad \begin{aligned} E_{-\gamma} &\sim [E_\alpha, [E_\alpha, E_{\beta^*}]] \quad (\text{by (4.42)}) \\ &= [E_\alpha, [E_{\alpha^*}, E_\beta]] \quad (\text{by (SR6)}) \end{aligned}$$

$$(4.52) \quad = [E_{\alpha^*}, [E_\alpha, E_\beta]] \quad (\text{by (SR5)}).$$

Then (4.48) follows from (4.51) and (SR5). We show (4.49). We have

$$\begin{aligned}
[E_{-\gamma}, E_\beta] &\sim [[E_{\alpha^*}, [E_\alpha, E_\beta]], E_\beta] \quad (\text{by (4.52)}) \\
&= [[E_{\alpha^*}, E_\beta], [E_\alpha, E_\beta]] \quad (\text{by (SR5)}) \\
&\sim n_\beta([E_{\alpha^*}, E_\alpha]) \quad (\text{by (4.13)}) \\
&= 0 \quad (\text{by (SR5)}),
\end{aligned}$$

which implies (4.49), as desired.

Case 3: $R^{\{\alpha, \beta\}}$ is $D_3^{(2)}$, $C^{(2)}(3)$ or $A^{(4)}(0, 4)$ (i.e., (iii) of (4.14)). By (4.42), we have

$$(4.53) \quad E_{-\gamma} \sim [E_\beta, E_{\alpha^*}].$$

By (4.53) and (SR8), we have (4.48). By (4.53) and (SR5), we have (4.49).

Case 4: $R^{\{\alpha, \beta\}}$ is $A_4^{(2)}$ or $B^{(1)}(0, 2)$ (i.e., (iv) of (4.14)). We have

$$(4.54) \quad E_{-\gamma} \sim [E_\beta, [E_\beta, E_{\alpha^*}]] \quad (\text{by (4.42)})$$

$$(4.55) \quad \sim [E_\beta, [E_\alpha, [E_\alpha, E_{\beta^*}]]] \quad (\text{by (SR6)}).$$

Then (4.49) follows from (4.54) and (SR5). We show (4.48). We have

$$\begin{aligned}
[E_{-\gamma}, E_\alpha] &\sim [[E_\beta, [E_\alpha, [E_\alpha, E_{\beta^*}]]], E_\alpha] \quad (\text{by (4.55)}) \\
&\sim [[E_\beta, E_\alpha], [E_\alpha, [E_\alpha, E_{\beta^*}]]] \quad (\text{by (SR5)}) \\
&\sim [[E_\beta, E_\alpha], [E_\beta, E_{\alpha^*}]] \quad (\text{by (SR6)}) \\
&\sim n_\beta([E_\alpha, [E_\beta, E_{\alpha^*}]]) \\
&\quad (\text{by (4.13) and } n_\beta([E_\beta, E_{\alpha^*}]) \sim [E_\beta, E_{\alpha^*}] \text{ (see (4.12))}) \\
&\sim n_\beta([E_\alpha, [E_\alpha, [E_\alpha, E_{\beta^*}]]]) \quad (\text{by (SR6)}) \\
&= 0 \quad (\text{by (SR5)}).
\end{aligned}$$

Then we have (4.48).

(Notice that $R^{\{\alpha, \beta\}}$ cannot be $G_2^{(1)}$ nor $D_4^{(3)}$ (see (4.40)).)

Step 5. (2) holds. By Steps 1-4 and Lemma 4.2, together with (1) of this lemma, we see that (2) holds. This completes the proof. \square

Lemma 4.6. *See (4.1)-(4.2) and (4.24)-(4.25). Let $\alpha, \beta \in \Pi$ be such that $\alpha \neq \beta$ and $(\alpha, \beta) = 0$. Then we have*

$$(4.56) \quad \mathfrak{g}^{\{\alpha, \beta\}} = \mathfrak{g}^{\{\alpha\}} + \mathfrak{g}^{\{\beta\}}$$

and

$$(4.57) \quad R^{\{\alpha, \beta\}} = R^{\{\alpha\}} \cup R^{\{\beta\}}.$$

Proof. By (SR1-3,5), $[\mathfrak{g}^{\{\alpha\}}, \mathfrak{g}^{\{\beta\}}] \subset \mathfrak{g}^{\{\alpha\}} + \mathfrak{g}^{\{\beta\}}$. Then we have (4.56). By (3.23), we have (4.57). \square

4.7 Easy lemma

Lemma 4.7. *Let \mathfrak{a} be a Lie superalgebra. Let \mathfrak{a}_i ($1 \leq i \leq r$) be subspaces of \mathfrak{a} . Then we have*

$$(4.58) \quad (\text{ada}_1) \cdots (\text{ada}_{r-1})(\mathfrak{a}_r) \subset \sum_{\sigma \in S_{r-1}} (\text{ada}_{\sigma(1)+1}) \cdots (\text{ada}_{\sigma(r-1)+1})(\mathfrak{a}_1),$$

where S_{r-1} is the symmetric group of degree $r-1$.

This can be proved easily by using induction on r . The proof of this lemma shall be given in Section 8.

4.8 Triangular decomposition

See (4.1)-(4.2). Let S be a non-empty subset of Π . Let

$$(4.59) \quad R^{S,+} = R^S \cap (\mathbb{Z}_+ S \oplus \mathbb{Z}a) \text{ and } R^{S,-} = R^S \cap (\mathbb{Z}_- S \oplus \mathbb{Z}a)$$

(see (4.24) for R^S). By (3.21), we see

$$(4.60) \quad R^S = R^{S,+} \cup R^{S,-}.$$

Let $m_S = \min\{k(\alpha) | \alpha \in S\}$. Define the sub-Lie superalgebras $\mathfrak{n}^{S,+}$, $\mathfrak{n}^{S,-}$, $\mathfrak{l}^{S,+}$ and $\mathfrak{l}^{S,-}$ of \mathfrak{g}^S (see (4.25)) by

$$\mathfrak{n}^{S,\pm} = \bigoplus_{\mu \in (\mathbb{Z}_\pm S \oplus \mathbb{Z}m_S a) \setminus \mathbb{Z}m_S a} \mathfrak{g}_\mu^S$$

and

$$\mathfrak{l}^{S,\pm} = \bigoplus_{r \in \mathbb{Z}_\pm \setminus \{0\}} \mathfrak{g}_{rm_S a}^S.$$

Let $\mathfrak{n}^\pm = \mathfrak{n}^{\Pi,\pm}$ and $\mathfrak{l}^\pm = \mathfrak{l}^{\Pi,\pm}$.

Lemma 4.8. *Keep the notation as above. We have:*

(1)

$$(4.61) \quad \mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{l}^+ \oplus \mathfrak{h} \oplus \mathfrak{l}^- \oplus \mathfrak{n}^-.$$

(2) \mathfrak{n}^+ (resp. \mathfrak{n}^-) is generated by $\mathfrak{n}^{\{\alpha\},+}$ (resp. $\mathfrak{n}^{\{\alpha\},-}$) with all $\alpha \in \Pi$. Namely, letting $\mathfrak{n}^{(1),\pm} = \bigoplus_{\alpha \in \Pi} \mathfrak{n}^{\{\alpha\},\pm}$ and $\mathfrak{n}^{(i),\pm} = [\mathfrak{n}^{(1),\pm}, \mathfrak{n}^{(i-1),\pm}]$ ($i \geq 2$), we have $\mathfrak{n}^\pm = \sum_{i=1}^{\infty} \mathfrak{n}^{(i),\pm}$.

$$(3) \mathfrak{l}^+ = \sum_{\alpha \in \Pi} \mathfrak{l}^{\{\alpha\},+} \text{ and } \mathfrak{l}^- = \sum_{\alpha \in \Pi} \mathfrak{l}^{\{\alpha\},-}.$$

Proof. (Strategy. Notice that \mathfrak{g} is generated by $\mathfrak{n}^{\{\alpha\},+}$ and $\mathfrak{n}^{\{\alpha\},-}$ with all $\alpha \in \Pi$, together with \mathfrak{h} . Using the commutator formulas (4.63) and (4.64), especially $[\mathfrak{n}^{\{\alpha\},+}, \mathfrak{n}^{\{\beta\},-}] = \{0\}$ ($\alpha \neq \beta$), we have

$$(4.62) \quad \mathfrak{g} = \mathfrak{h} \oplus \left(\sum_{i=1}^{\infty} \mathfrak{n}^{(i),+} \right) \oplus \left(\sum_{i=1}^{\infty} \mathfrak{n}^{(i),-} \right) \oplus \left(\sum_{\alpha \in \Pi} (\mathfrak{l}^{\{\alpha\},+} \oplus \mathfrak{l}^{\{\alpha\},-}) \right).$$

Then (1)-(3) easily follow from (4.62).)

We proceed in steps.

Step 1. \mathfrak{g} is generated by \mathfrak{h} , $\mathfrak{n}^{(1),+}$ and $\mathfrak{n}^{(1),-}$. This is clear since $E_{\pm\alpha}$, $E_{\pm\alpha^*} \in \mathfrak{n}^{\{\alpha\},\pm}$ for $\alpha \in \Pi$.

Step 2. Commutator formulas for each $\alpha \in \Pi$. Let $\alpha \in \Pi$. By (4.30), we have

$$(4.63) \quad \begin{cases} [\mathfrak{n}^{\{\alpha\},\pm}, \mathfrak{n}^{\{\alpha\},\pm}] \subset \mathfrak{n}^{\{\alpha\},\pm}, \\ [\mathfrak{n}^{\{\alpha\},+}, \mathfrak{n}^{\{\alpha\},-}] \subset \mathfrak{n}^{\{\alpha\},+} \oplus \mathfrak{l}^{\{\alpha\},+} \oplus \mathfrak{h} \oplus \mathfrak{l}^{\{\alpha\},-} \oplus \mathfrak{n}^{\{\alpha\},-} (= \mathfrak{g}^{\{\alpha\}}), \\ [\mathfrak{n}^{\{\alpha\},\pm}, \mathfrak{l}^{\{\alpha\},+} \oplus \mathfrak{l}^{\{\alpha\},-}] \subset \mathfrak{n}^{\{\alpha\},\pm} \end{cases}$$

(notice that $[\mathfrak{g}_{\pm\alpha+r_1k(\alpha)a}^{\{\alpha\}}, \mathfrak{g}_{\mp 2\alpha+r_2k(\alpha)a}^{\{\alpha\}}] \subset \mathfrak{n}^{\{\alpha\},\mp}$ for $r_1 \in \mathbb{Z}$ and $r_2 \in g(\alpha)$).

Step 3. Commutator formulas for $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$. Let $\alpha, \beta \in \Pi$ be such that $\alpha \neq \beta$. By (4.31), (4.39) and (4.56), we see that if $\nu \in \{\alpha, \beta\}$ and $\mu \in R^{\{\nu\}}$, then $\mathfrak{g}_\mu^{\{\nu\}} = \mathfrak{g}_\mu^{\{\alpha, \beta\}}$. Then, by (4.38), (4.56) and (4.60), we have

$$(4.64) \quad \begin{cases} [\mathfrak{n}^{\{\alpha\},+}, \mathfrak{n}^{\{\beta\},-}] = \{0\}, \\ [\mathfrak{n}^{\{\alpha\},\pm}, \mathfrak{l}^{\{\beta\},+} \oplus \mathfrak{l}^{\{\beta\},-}] \subset \mathfrak{n}^{\{\alpha\},\pm}. \end{cases}$$

Step 4. (1)-(3) hold. Let $\mathfrak{i}^\pm = \sum_{\alpha \in \Pi} \mathfrak{l}^{\{\alpha\},\pm}$. By (4.63) and (4.64), we have

$$(4.65) \quad [\mathfrak{n}^{(1),+}, \mathfrak{n}^{(1),-} \oplus (\mathfrak{i}^+ \oplus \mathfrak{h} \oplus \mathfrak{i}^-)] \subset \mathfrak{n}^{(1),+} \oplus \mathfrak{i}^+ \oplus \mathfrak{h} \oplus \mathfrak{i}^- \oplus \mathfrak{n}^{(1),-}.$$

Then, for $i \in \mathbb{N}$, we have

$$(4.66) \quad \begin{aligned} & [\mathfrak{n}^{(1),-}, \mathfrak{n}^{(i),+}] \\ & \subset (\text{adn}^{(1),+})^i(\mathfrak{n}^{(1),-}) \quad (\text{by Lemma 4.7}) \\ & = (\text{adn}^{(1),+})^{i-1}([\mathfrak{n}^{(1),+}, \mathfrak{n}^{(1),-}]) \\ & \subset \left(\sum_{j=1}^i \mathfrak{n}^{(j),+} \right) \oplus \mathfrak{i}^+ \oplus \mathfrak{h} \oplus \mathfrak{i}^- \oplus \mathfrak{n}^{(1),-} \quad (\text{by (4.65)}). \end{aligned}$$

Similarly we have

$$(4.67) \quad [\mathfrak{n}^{(1),+}, \mathfrak{n}^{(i),-}] \subset \mathfrak{n}^{(1),+} \oplus \mathfrak{l}^+ \oplus \mathfrak{h} \oplus \mathfrak{l}^- \oplus \left(\sum_{j=1}^i \mathfrak{n}^{(j),-} \right).$$

By (4.66)-(4.67) (and (SR1-3)) and Step 1, we have

$$(4.68) \quad \mathfrak{g} = \left(\sum_{j=1}^{\infty} \mathfrak{n}^{(j),+} \right) \oplus \mathfrak{l}^+ \oplus \mathfrak{h} \oplus \mathfrak{l}^- \oplus \left(\sum_{j=1}^{\infty} \mathfrak{n}^{(j),-} \right).$$

(Hence we have (4.62).) By (4.68), we see that (1)-(3) hold. This completes the proof. \square

Lemma 4.9. *See (4.1)-(4.2) and (4.24). We have:*

(1)

$$(4.69) \quad \begin{cases} \{\lambda \in \mathcal{E} \mid \dim \mathfrak{g}_\lambda \neq 0\} \subset (\mathbb{Z}_+ \Pi \oplus \mathbb{Z}a) \cup (\mathbb{Z}_- \Pi \oplus \mathbb{Z}a) \\ \text{and } \mathfrak{g}_0 = \mathfrak{h}. \end{cases}$$

(2) *For each $\alpha \in \Pi$, it follows that*

$$(4.70) \quad \dim \mathfrak{g}_\mu = 1 \text{ and } \mathfrak{g}_\mu \subset \mathfrak{g}(p(\mu)) \text{ for all } \mu \in R^{\{\alpha\}}$$

and that

$$(4.71) \quad \dim \mathfrak{g}_\lambda = 0 \text{ for all } \lambda \in (\mathbb{Z}\alpha \oplus \mathbb{Z}a) \setminus (R^{\{\alpha\}} \cup \mathbb{Z}a).$$

Proof. (*Strategy.* We use Lemma 4.8 (1)-(2) and the fact that for $\mu \in R^{\{\alpha\},\pm}$, $\mathfrak{g}_\mu \subset \mathfrak{n}^\pm$.)

(1) This is clear from (4.61) (see also (4.3)).

(2) By Lemma 4.8 (2), we have

$$\left(\bigoplus_{\lambda \in (\pm N\alpha) \oplus \mathbb{Z}a} \mathfrak{g}_\lambda \right) \cap \mathfrak{n}^\pm = \mathfrak{n}^{\{\alpha\},\pm}.$$

Then, by (4.61), we have

$$(4.72) \quad \bigoplus_{\lambda \in (\mathbb{Z}\alpha \oplus \mathbb{Z}a) \setminus \mathbb{Z}a} \mathfrak{g}_\lambda = \mathfrak{n}^{\{\alpha\},+} \oplus \mathfrak{n}^{\{\alpha\},-} = \bigoplus_{\mu \in R^{\{\alpha\}}} \mathfrak{g}_\mu^{\{\alpha\}}.$$

Then (2) follows from (4.31) and (4.72). This completes the proof. \square

Lemma 4.10. See (4.1)-(4.2). We have:

(1)

$$(4.73) \quad \mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\mu \in R} \mathfrak{g}_\mu \right) \bigoplus \left(\bigoplus_{(m,r) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \mathfrak{g}_{m\delta+ra} \right).$$

(2) $\dim \mathfrak{h} = l + 4$.

(3) $\dim \mathfrak{g}_\mu = 1$ and $\mathfrak{g}_\mu \subset \mathfrak{g}(p(\mu))$ for all $\mu \in R$.

Namely \mathfrak{g} has the same properties as in Theorem 1.1 (1).

Proof. (*Strategy.* We can prove this lemma easily using (4.11) and Lemmas 4.1, 4.4 and 4.9.)

(2) This is clear from Lemma 4.1.

(3) Let $\mu \in R$. By (4.32), there exist $\alpha \in \Pi$, $\nu \in R^{\{\alpha\}}$ and $w \in W_\Pi$ such that $p(\mu) = p(\nu)$ and $w(\mu) = \nu$. By (4.11) and (4.70), for $i \in \{0, 1\}$, we have $\dim(\mathfrak{g}_\mu \cap \mathfrak{g}(i)) = \dim(\mathfrak{g}_\nu \cap \mathfrak{g}(i)) = \delta_{i,p(\nu)} = \delta_{i,p(\mu)}$. Then, by (3.27), we have (3).

(1) This follows easily from (4.11), (4.33), (4.71) and (4.69). \square

4.9 Proof of Theorem 1.1 (1) and Theorem 3.2

Here we give proofs of Theorem 1.1 (1) and Theorem 3.2.

Proof of Theorem 1.1 (1). (*Strategy.* This follows from Lemma 4.10 and Theorem 3.1. (See Subsection 1.8 for entire strategy.)) By Theorem 3.1, we see that R can be identified with $R(\Pi, k, g)$ of (3.15) for some $(\Pi, k, g) \in T_l^{\text{nrsm}}$ (see (3.28) for T_l^{nrsm}). Then the statement follows from Lemma 4.10. (Notice that the definitions of $\mathfrak{g}(\Pi, k, g)$ given by Definition 1.3 and (3.26) are the same.) \square

Proof of Theorem 3.2. (*Strategy.* This follows from (4.11) and the root space decomposition (4.73) of $\mathfrak{g}(\Pi, k, g)$.) If $(\Pi, k, g) \in T_l^{\text{rm}}$ (cf. (3.28)), the theorem follows from [13, (6.4)-(6.6)]. Assume $(\Pi, k, g) \in T_l^{\text{nrsm}}$. Let $R = R(\Pi, k, g)$ be of (3.15). Let $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$ be of (3.26). Recall (AX1-5) from Definition 1.1. By (3.21), we see that $\dim V^0 = 2$ and R satisfy (AX1) and (AX2). Since $\pi_a(R)$ is an affine root system (see (3.16)), we see that R satisfy (AX3) and (AX5). By (4.73) and Lemma 4.10 (3), we have

$$(4.74) \quad R = \{\mu \in V \mid (\mu, \mu) \neq 0, \dim \mathfrak{g}_\mu \neq 0\}.$$

Then, by (4.74), Lemma 4.4 (4) and (4.11), we see that R satisfies (AX4). Hence R is an elliptic root system. By (3.16), (3.21) and (3.22), we see that $\Pi \cup \{a\}$ is a fundamental-set of R . This completes the proof. \square

5 Examples of representations (Proof of Lemma 4.1)

The purpose of this section is to prove Lemma 4.1.

5.1 Strategy of the proof of Lemma 4.1

We prove Lemma 4.1 by case-by-case checking. In Lemmas 5.5-5.9 for each $(\Pi, k, g) \in T_l^{\text{nrsm}}$, we shall explicitly give a Lie superalgebra \mathbb{L} and a homomorphism $\psi_i : \mathfrak{g}(\Pi, k, g) \rightarrow \mathbb{L}$ such that $\dim \psi_i(\mathfrak{h}(\Pi, k, g)) = l + 4$. Let $(\Pi, k, g) \in T_l^{\text{nrsm}}$. Let $A(\Pi)$ be as in Definition 3.1 (1) (1-ii). Then $A(\Pi)$ is $D_{l+1}^{(2)}$, $A_{2l}^{(2)}$, $B_l^{(1)}$ or $C_2^{(1)}$ (see (3.29)). If $A(\Pi)$ is $D_{l+1}^{(2)}$, then $\mathbb{L} = \bar{\mathfrak{G}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\bar{v} \oplus \mathbb{C}\bar{w}$, where a Lie superalgebra $\bar{\mathfrak{G}}$ is $\mathfrak{sl}(m|r)^{(i)}$ or $\mathfrak{osp}(m|r)^{(i)}$. If $A(\Pi)$ is $A_{2l}^{(2)}$ or $B_l^{(1)}$, then $\mathbb{L} = \mathfrak{g}(\Pi', k', g')$ for some $(\Pi', k', g') \in T_l^{\text{nrsm}}$ with $A(\Pi') = D_{l+1}^{(2)}$. If $A(\Pi)$ is $C_2^{(1)}$, then $\mathbb{L} = \mathfrak{g}(\Pi', k', g')$ for some $(\Pi', k', g') \in T_l^{\text{nrsm}}$ with $A(\Pi) = B_3^{(1)}$ or $\mathbb{L} = F(4)^{(1)} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\bar{v} \oplus \mathbb{C}\bar{w}$. In the proofs of Lemmas 5.5-5.9, we check that $\psi_i(h_\sigma)$ ($\sigma \in \mathcal{E}(\Pi, k, g)$) and $\psi_i(E_\mu)$ ($\mu \in \mathcal{B}$) satisfy (SR1-9). We can also use the facts (5.1)-(5.2) below:

$$(5.1) \quad \left\{ \begin{array}{l} \text{As for (SR5), since } \psi_i(E_\mu) \text{ are locally ad-nilpotent} \\ \text{(cf. Lemma 5.3)}, \text{ we only need to check that} \\ [\psi_i(E_\mu), \psi_i(E_\nu)] = 0 \text{ for } \mu, \nu \in \mathcal{B} \text{ with } \mu \neq \nu \text{ and } (\mu, \nu) > 0 \\ \text{and for } \mu \in \mathcal{B}_+ \text{ and } \nu \in -\mathcal{B}_+ \text{ with } (\mu, \nu) = 0 \end{array} \right.$$

(this can be seen by the same formulas as (4.6)).

$$(5.2) \quad \left\{ \begin{array}{l} \text{As for (SR4), to check } [\psi_i(E_\mu), \psi_i(E_{-\mu})] = \psi_i(h_{\mu^\vee}) \text{ for } \mu \in \mathcal{B}_+ \setminus \Pi, \\ \text{we can avoid some calculation (see Remark 8.1).} \end{array} \right.$$

5.2 Contragredient Lie superalgebras

We first recall the definition of the contragredient Lie superalgebras [9]. (See also Subsection 1.5 for the terminology.) Let \bar{I} be a finite set. Let \bar{I}^{odd} be a subset of \bar{I} . Define a map $\bar{p} : \bar{I} \rightarrow \{0, 1\}$ by $\bar{p}(i) = 1$ ($i \in \bar{I}^{\text{odd}}$) and $\bar{p}(j) = 0$ ($j \in \bar{I} \setminus \bar{I}^{\text{odd}}$). Let $\bar{A} = (\bar{a}_{ij})_{i,j \in \bar{I}}$ be an $|\bar{I}| \times |\bar{I}|$ matrix such that there exist $\bar{e}_i \in \mathbb{C} \setminus \{0\}$ ($i \in \bar{I}$) satisfying the condition that $\bar{D}^{-1}\bar{A}$ is a symmetric matrix, where \bar{D} is the diagonal matrix $\text{diag}(\bar{e}_1, \dots, \bar{e}_{|\bar{I}|})$. Such \bar{A} is called a symmetrizable matrix. For the triple $(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$ of these \bar{A} , \bar{I} and \bar{I}^{odd} , we define a \mathbb{C} -Lie superalgebra $\bar{\mathfrak{G}} = \bar{\mathfrak{G}}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$ in the following way. We define a \mathbb{C} -Lie superalgebra $\bar{\mathfrak{G}}' = \bar{\mathfrak{G}}'(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$ by generators:

$$\bar{h}'_i, \bar{t}'_i, \bar{E}'_i, \bar{F}'_i \quad (i \in \bar{I})$$

with parities:

$$p(\bar{h}'_i) = p(\bar{t}'_i) = 0, \quad p(\bar{E}'_i) = p(\bar{F}'_i) = \bar{p}(i)$$

and defining relations:

$$\begin{cases} [\bar{h}'_i, \bar{h}'_j] = [\bar{h}'_i, \bar{t}'_j] = [\bar{t}'_i, \bar{t}'_j] = 0, \\ [\bar{h}'_i, \bar{E}'_j] = \bar{a}_{ij}\bar{E}'_j, [\bar{t}'_i, \bar{E}'_j] = \delta_{ij}\bar{E}'_j, \\ [\bar{h}'_i, \bar{F}'_j] = -\bar{a}_{ij}\bar{F}'_j, [\bar{t}'_i, \bar{F}'_j] = -\delta_{ij}\bar{F}'_j, \\ [\bar{E}'_i, \bar{F}'_j] = \delta_{ij}\bar{h}'_i. \end{cases}$$

Let $\bar{\mathfrak{H}}'$ be the sub-Lie superalgebra of $\bar{\mathfrak{G}}'$ generated by \bar{h}'_i, \bar{t}'_i ($i \in \bar{I}$). By a well-known argument (see [8, §1.3]), we see that $\{\bar{h}'_i, \bar{t}'_i | i \in \bar{I}\}$ is a basis of $\bar{\mathfrak{H}}'$, i.e., $\dim \bar{\mathfrak{H}}' = 2|\bar{I}|$. Let \mathfrak{r} be the ideal of $\bar{\mathfrak{G}}'$ maximal among the ideals \mathfrak{r}' such that $\mathfrak{r}' \cap \bar{\mathfrak{H}}' = \{0\}$. Then we define $\bar{\mathfrak{G}} = \bar{\mathfrak{G}}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$ as the quotient Lie superalgebra $\bar{\mathfrak{G}}'/\mathfrak{r}$. The Lie superalgebra $\bar{\mathfrak{G}}$ is a *contragredient Lie superalgebra* (see [9]). Let $\bar{\pi} : \bar{\mathfrak{G}}' \rightarrow \bar{\mathfrak{G}}$ be the canonical map. We denote $\bar{\pi}(\bar{\mathfrak{H}}')$, $\bar{\pi}(\bar{h}'_i)$, $\bar{\pi}(\bar{t}'_i)$, $\bar{\pi}(\bar{E}'_i)$ and $\bar{\pi}(\bar{F}'_i)$ by $\bar{\mathfrak{H}}$, \bar{h}_i , \bar{t}_i , \bar{E}_i and \bar{F}_i respectively.

5.3 Properties of contragredient Lie superalgebras

By a well-known argument (see [10]), we have the following.

Lemma 5.1. $\dim \bar{\mathfrak{H}} = 2|\bar{I}|$, i.e., $\{\bar{h}_i, \bar{t}_i | i \in \bar{I}\}$ is a basis of $\bar{\mathfrak{H}}$.

Let $\bar{\mathfrak{E}} = \bar{\mathfrak{H}}^*$ (the dual space of $\bar{\mathfrak{H}}$). Define a basis $\{\bar{\alpha}_i, \bar{\gamma}_i | i \in \bar{I}\}$ of $\bar{\mathfrak{E}}$ by $\bar{\alpha}_i(\bar{h}_j) = \bar{a}_{ji}$, $\bar{\alpha}_i(\bar{t}_j) = \bar{\gamma}_i(\bar{h}_j) = \delta_{ij}$ and $\bar{\gamma}_i(\bar{t}_j) = 0$. Define a non-degenerate

symmetric bilinear form $(,) : \bar{\mathfrak{E}} \times \bar{\mathfrak{E}} \rightarrow \mathbb{C}$ by $(\bar{\alpha}_i, \bar{\alpha}_j) = \bar{a}_{ij}/\bar{\epsilon}_i$, $(\bar{\gamma}_i, \bar{\alpha}_j) = \delta_{ij}/\bar{\epsilon}_i$ and $(\bar{\gamma}_i, \bar{\gamma}_j) = 0$. For $\bar{\sigma} \in \bar{\mathfrak{E}}$, denote by $h_{\langle \bar{\sigma} \rangle}$ the element of $\bar{\mathfrak{H}}$ satisfying the property that $\bar{\tau}(h_{\langle \bar{\sigma} \rangle}) = (\bar{\tau}, \bar{\sigma})$ ($\bar{\tau} \in \bar{\mathfrak{H}}$). Then we have $\bar{h}_i = \bar{\epsilon}_i h_{\langle \bar{\alpha}_i \rangle}$ and $\bar{t}_i = \bar{\epsilon}_i h_{\langle \bar{\gamma}_i \rangle}$.

The following is also proved by a well-known argument (see [8, Theorem 2.2]).

Lemma 5.2. *There exists a super-symmetric invariant form $(,) : \bar{\mathfrak{G}} \times \bar{\mathfrak{G}} \rightarrow \mathbb{C}$ such that $(h_{\langle \bar{\sigma} \rangle}, h_{\langle \bar{\tau} \rangle}) = (\bar{\sigma}, \bar{\tau})$ and $(\bar{E}_i, \bar{F}_j) = \delta_{ij}\bar{\epsilon}_i$.*

By the same argument as in the proof of [17, Theorem 4.1.1], we have the following.

Lemma 5.3. *We have the following:*

- (1) $(\text{ad } \bar{E}_i)^{1-\bar{a}_{ij}} \bar{E}_j = 0$ if $\bar{p}(i) = 0$, $\bar{a}_{ii} = 2$ and $\bar{a}_{ij} \in \mathbb{Z}_-$ or if $\bar{p}(i) = 1$, $\bar{a}_{ii} = 2$ and $\bar{a}_{ij} \in 2\mathbb{Z}_-$.
- (2) $[\bar{E}_i, \bar{E}_j] = 0$ if $\bar{a}_{ij} = \bar{a}_{ji} = 0$.
- (3) $[\bar{E}_j, [[\bar{E}_i, \bar{E}_j], \bar{E}_r]] = 0$ if $i \neq j \neq r \neq i$, $\bar{p}(j) = 1$, $\bar{a}_{jj} = \bar{a}_{ir} = \bar{a}_{ri} = 0$ and $\bar{a}_{ji} + \bar{a}_{jr} = 0$.
- (4) $[[\bar{E}_i, \bar{E}_j], \bar{E}_r] = [[\bar{E}_i, \bar{E}_r], \bar{E}_j]$ if $i \neq j \neq r \neq i$, $\bar{p}(j) = \bar{p}(r) = 1$, $\bar{a}_{ij} = \bar{a}_{ji} = \bar{a}_{ir} = \bar{a}_{ri} = -1$ and $\bar{a}_{jr} = \bar{a}_{rj} = 2$.
- (5) *The same formulas as those in (1)-(4) with \bar{F}_i 's in place of \bar{E}_i 's hold.*

Let $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomial algebra. Let

$$(5.3) \quad \mathfrak{L} = \mathfrak{L}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}}) = \bar{\mathfrak{G}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\bar{v} \oplus \mathbb{C}\bar{w}.$$

We view \mathfrak{L} as a Lie superalgebra in the following way (cf. Subsection 1.5). If $X \in \bar{\mathfrak{G}}(i)$, let $X \otimes t^n \in \mathfrak{L}(i)$; let $\bar{v}, \bar{w} \in \mathfrak{L}(0)$ (cf. (1.26)). The Lie super-bracket of \mathfrak{L} is given by

$$\begin{aligned} & [X \otimes t^m + a_1\bar{v} + b_1\bar{w}, Y \otimes t^n + a_2\bar{v} + b_2\bar{w}] \\ &= [X, Y] \otimes t^{m+n} + m\delta_{m+n,0}(X, Y)\bar{v} + b_1nY \otimes t^n - b_2mX \otimes t^m. \end{aligned}$$

We shall also denote by $(,)$ the super-symmetric invariant form on \mathfrak{L} defined by

$$(5.4) \quad (X \otimes t^m + a_1\bar{v} + b_1\bar{w}, Y \otimes t^n + a_2\bar{v} + b_2\bar{w}) = \delta_{m+n,0}(X, Y) + a_1b_2 + b_1a_2.$$

5.4 Notation $T_l^{\text{norm}}(A)$ and numbering of $\alpha_0, \dots, \alpha_l$

Let $l \geq 2$. For A which is an $(l+1) \times (l+1)$ generalized Cartan matrix of affine-type, we denote by $T_l^{\text{norm}}(A)$ the subset of T_l^{norm} (see (3.28)) formed by the elements $(\Pi, k, g) \in T_l^{\text{norm}}$ such that $A(\Pi) = A$ (see Definition 3.1 (1) (1-ii) for $A(\Pi)$). By (3.29), we have

$$(5.5) \quad T_l^{\text{norm}} = \begin{cases} T_l^{\text{norm}}(B_l^{(1)}) \cup T_l^{\text{norm}}(A_{2l}^{(2)}) \cup T_l^{\text{norm}}(D_{l+1}^{(2)}) & \text{if } l \geq 3, \\ T_2^{\text{norm}}(C_2^{(1)}) \cup T_2^{\text{norm}}(A_4^{(2)}) \cup T_2^{\text{norm}}(D_3^{(2)}) & \text{if } l = 2. \end{cases}$$

Let $(\Pi = \{\alpha_0, \dots, \alpha_l\}, k, g) \in T_l^{\text{norm}}$ with $l \geq 2$. From now on, we assume that $(\alpha_i, \alpha_i) \leq (\alpha_{i+1}, \alpha_{i+1})$ and $(\alpha_i, \alpha_{i+1}) \neq 0$ for $1 \leq i \leq l-1$. Then $(\alpha_0, \alpha_0) \geq (\alpha_1, \alpha_1)$ (see (3.19)). We define a total order on $\{\emptyset, 2\mathbb{Z} + 1, \mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}\}$ by $4\mathbb{Z} < 2\mathbb{Z} < \mathbb{Z} < 2\mathbb{Z} + 1 < \emptyset$. We also assume that if $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$ and $i < j$, then $g(\alpha_i) > g(\alpha_j)$ or $g(\alpha_i) = g(\alpha_j)$ and $k(\alpha_i) \geq k(\alpha_j)$. Set

$$(5.6) \quad \begin{cases} \Pi_{\text{sh}} := \Pi \cap \pi_a^{-1}(\pi_a(\Pi)_{\text{sh}}), \Pi_{\text{lg}} := \Pi \cap \pi_a^{-1}(\pi_a(\Pi)_{\text{lg}}) \text{ and} \\ \Pi_{\text{ex}} := \Pi \cap \pi_a^{-1}(\pi_a(\Pi)_{\text{ex}}) \end{cases}$$

(see also (1.47) and (3.16)).

5.5 Unfolding for $D_{l+1}^{(2)}$

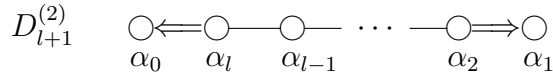


Fig. 4.1

In this subsection, we always let $(\Pi, k, g) \in T_l^{\text{norm}}(D_{l+1}^{(2)})$ with $l \geq 2$. Then $\Pi_{\text{sh}} = \{\alpha_0, \alpha_1\}$, $\Pi_{\text{lg}} = \{\alpha_i \mid 2 \leq i \leq l\}$ and $\Pi_{\text{ex}} = \emptyset$ (cf. (5.6)). Moreover, we see that $k(\Pi)$ is $\{1\}$ or $\{1, 2\}$ (see (3.13)). Notice that $g(\beta) = \emptyset$ for $\beta \in \Pi_{\text{lg}}$ and that $k(\beta_1) = k(\beta_2)$ for $\beta_1, \beta_2 \in \Pi_{\text{lg}}$. Moreover, we notice that for $\alpha \in \Pi_{\text{sh}}$ and $\beta \in \Pi_{\text{lg}}$, it follows that $(k(\alpha), g(\alpha), k(\beta))$ is $(1, \emptyset, 1)$, $(1, \emptyset, 2)$, $(2, \emptyset, 2)$, $(1, 2\mathbb{Z} + 1, 1)$, $(2, 2\mathbb{Z} + 1, 2)$, $(1, \mathbb{Z}, 1)$, $(2, \mathbb{Z}, 2)$, $(1, 2\mathbb{Z}, 1)$, $(1, 2\mathbb{Z}, 2)$, $(2, 2\mathbb{Z}, 2)$ or $(1, 4\mathbb{Z}, 2)$ (see also Table 4.1). Let β denote an element of Π_{lg} and define

a map $\vartheta : \Pi_{\text{sh}} \rightarrow \{1, 2, 3\}$ by

$$\vartheta(\alpha) = \begin{cases} 1 & \text{if } g(\alpha) = \emptyset \text{ and } k(\alpha) = k(\beta), \\ 1 & \text{if } g(\alpha) = \mathbb{Z} \text{ (this implies } k(\alpha) = k(\beta)), \\ 2 & \text{if } g(\alpha) = \emptyset \text{ and } 2k(\alpha) = k(\beta)(= 2), \\ 2 & \text{if } g(\alpha) = 2\mathbb{Z} \text{ and } 2k(\alpha) = k(\beta)(= 2), \\ 3 & \text{if } g(\alpha) = 2\mathbb{Z} \text{ and } k(\alpha) = k(\beta), \\ 3 & \text{if } g(\alpha) = 2\mathbb{Z} + 1 \text{ (this implies } k(\alpha) = k(\beta)), \\ 3 & \text{if } g(\alpha) = 4\mathbb{Z} \text{ (this implies } 2k(\alpha) = k(\beta)(= 2)). \end{cases}$$

We use ϑ in order to define k^\vee below.

We define a map $k^\vee : \Pi \rightarrow \{1, 2, 3, 4\}$ as follows.

(i) If $\vartheta(\alpha_0) = 3$ or $\vartheta(\alpha_1) = 3$ (i.e., $3 \in \vartheta(\Pi_{\text{sh}})$), let

$$(5.7) \quad k^\vee(\alpha') = \begin{cases} 2 & \text{if } \alpha' \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha') = 1, \\ 4 & \text{if } \alpha' \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha') = 2, \\ 2 & \text{if } \alpha' \in \Pi_{\text{sh}}, \vartheta(\alpha') = 3 \text{ and } g(\alpha') \in \{2\mathbb{Z}, 2\mathbb{Z} + 1\}, \\ 3 & \text{if } \alpha' \in \Pi_{\text{sh}}, \vartheta(\alpha') = 3 \text{ and } g(\alpha') = 4\mathbb{Z}, \\ 2 & \text{if } \alpha' \in \Pi_{\text{lg}}. \end{cases}$$

(ii) If $1 \leq \vartheta(\alpha_0), \vartheta(\alpha_1) \leq 2$ (i.e., $3 \notin \vartheta(\Pi_{\text{sh}})$), let

$$k^\vee(\alpha') = \begin{cases} 1 & \text{if } \alpha' \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha') = 1, \\ 2 & \text{if } \alpha' \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha') = 2, \\ 1 & \text{if } \alpha' \in \Pi_{\text{lg}}. \end{cases}$$

Remark 5.1. The situation $4 \in k^\vee(\Pi)$ occurs only when $\vartheta(\Pi_{\text{sh}}) = \{2, 3\}$ (see (5.7)). More precisely, for $\alpha \in \Pi$, $k^\vee(\alpha) = 4$ holds if and only if $\alpha \in \Pi_{\text{sh}}$, $\vartheta(\alpha) = 2$ and $\vartheta(\alpha') = 3$ for $\alpha' \in \Pi_{\text{sh}} \setminus \{\alpha\}$. See also Table 4.1 and Fig. 4.10 of Remark 5.2 and (v-1,2) of Example 5.1.

Define a subset $\widehat{I} = \widehat{I}(\Pi, k, g)$ of $\Pi \times \{1, 2, 3, 4\}$ by

$$\widehat{I} = \widehat{I}(\Pi, k, g) = \{(\alpha, x) \in \Pi \times \{1, 2, 3, 4\} \mid 1 \leq x \leq k^\vee(\alpha)\}.$$

We define an $|\widehat{I}| \times |\widehat{I}|$ matrix $\widehat{A} = \widehat{A}(\Pi, k, g) = (\widehat{a}_{(\alpha,x),(\beta,y)})_{(\alpha,x),(\beta,y) \in \widehat{I}}$ in the following way (see also Remark 5.2 and Example 5.1). For $\alpha, \beta \in \Pi$, we let $\widehat{A}_{\alpha,\beta}$ denote the $k^\vee(\alpha) \times k^\vee(\beta)$ submatrix $(\widehat{a}_{(\alpha,x),(\beta,y)})_{1 \leq x \leq k^\vee(\alpha), 1 \leq y \leq k^\vee(\beta)}$ of \widehat{A} . For $r \in \mathbb{N}$, let $e(r) = (e(r)_{ij})_{1 \leq i,j \leq r}$ denote the $r \times r$ identity matrix, i.e.,

$e(r)_{ii} = 1$ and $e(r)_{ij} = 0$ ($i \neq j$).

(i) Let $\alpha \in \Pi (= \Pi_{\text{sh}} \cup \Pi_{\text{lg}})$. Then we define

$$\widehat{A}_{\alpha,\alpha} = \begin{cases} 2e(k^\vee(\alpha)) & \text{if } \alpha \in \Pi_{\text{lg}}, \\ 2e(k^\vee(\alpha)) & \text{if } \alpha \in \Pi_{\text{sh}} \text{ and } g(\alpha) \in \{\emptyset, \mathbb{Z}\}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } \alpha \in \Pi_{\text{sh}}, g(\alpha) = 2\mathbb{Z} \text{ and } \vartheta(\alpha) = 3, \\ \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} & \text{if } \alpha \in \Pi_{\text{sh}}, g(\alpha) = 2\mathbb{Z}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 2, \\ \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} & \text{if } \alpha \in \Pi_{\text{sh}}, g(\alpha) = 2\mathbb{Z}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 4, \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} & \text{if } \alpha \in \Pi_{\text{sh}} \text{ and } g(\alpha) = 2\mathbb{Z} + 1, \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix} & \text{if } \alpha \in \Pi_{\text{sh}} \text{ and } g(\alpha) = 4\mathbb{Z}. \end{cases}$$

(ii) Assume $\alpha, \beta \in \Pi$ to be such that $\alpha \neq \beta$ and $(\alpha, \beta) = 0$. Then we define both $\widehat{A}_{\alpha,\beta}$ and $\widehat{A}_{\beta,\alpha}$ to be the zero matrices.

(iii) Assume $\beta_1, \beta_2 \in \Pi_{\text{lg}}$ to be such that $\beta_1 \neq \beta_2$ and $(\beta_1, \beta_2) \neq 0$. Notice $k^\vee(\beta_1) = k^\vee(\beta_2)$. Then we let $\widehat{A}_{\beta_1,\beta_2} = \widehat{A}_{\beta_2,\beta_1} = -e(k^\vee(\beta_1))$.

(iv) Assume $\alpha \in \Pi_{\text{sh}}$ and $\beta \in \Pi_{\text{lg}}$ to be such that $(\alpha, \beta) \neq 0$. Then $\widehat{A}_{\alpha,\beta}$ and $\widehat{A}_{\beta,\alpha}$ are defined by the following.

(iv-i) Assume $\vartheta(\alpha) = 1$. Notice $k^\vee(\alpha) = k^\vee(\beta) \in \{1, 2\}$. Then we let $\widehat{A}_{\alpha,\beta} = -2e(k^\vee(\alpha))$ and $\widehat{A}_{\beta,\alpha} = -e(k^\vee(\alpha))$.

(iv-ii) Assume $\vartheta(\alpha) = 2$. Notice that $(k^\vee(\alpha), k^\vee(\beta))$ is $(2, 1)$ or $(4, 2)$. We let

$$\widehat{A}_{\beta,\alpha} = \begin{cases} \begin{pmatrix} -1 & -1 \end{pmatrix} & \text{if } (k^\vee(\alpha), k^\vee(\beta)) = (2, 1), \\ \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} & \text{if } (k^\vee(\alpha), k^\vee(\beta)) = (4, 2), \end{cases}$$

and we let $\widehat{A}_{\alpha,\beta} = {}^t(\widehat{A}_{\beta,\alpha})$ (the transpose matrix of $\widehat{A}_{\beta,\alpha}$).

(iv-iii) Assume $\vartheta(\alpha) = 3$. Notice $k^\vee(\alpha) \in \{2, 3\}$ and $k^\vee(\beta) = 2$. Then we let

$$\widehat{A}_{\beta,\alpha} = \begin{cases} -e(2) & \text{if } (k^\vee(\alpha), k^\vee(\beta)) = (2, 2), \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} & \text{if } (k^\vee(\alpha), k^\vee(\beta)) = (3, 2), \end{cases}$$

and $\widehat{A}_{\alpha,\beta} = {}^t(\widehat{A}_{\beta,\alpha})$.

Lemma 5.4. *The matrix \widehat{A} is symmetrizable.*

This can be checked directly.

We define a subset \widehat{I}^{odd} of \widehat{I} by

$$\widehat{I}^{\text{odd}} = \{(\alpha, x) \in \widehat{I} \mid \widehat{a}_{(\alpha,x),(\alpha,x)} = 0\} \cup \{(\alpha, x) \in \widehat{I} \mid g(\alpha) = \mathbb{Z}\}.$$

Now we define the Lie superalgebra $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}(\Pi, k, g)$ by

$$(5.8) \quad \widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}(\Pi, k, g) = \mathfrak{L}(\widehat{A}, \widehat{I}, \widehat{I}^{\text{odd}})$$

(see (5.3) for $\mathfrak{L}(\widehat{A}, \widehat{I}, \widehat{I}^{\text{odd}})$).

Remark 5.2. *Dynkin diagrams of $\bar{\mathfrak{G}}(\widehat{A}(\Pi, k, g), \widehat{I}(\Pi, k, g), \widehat{I}^{\text{odd}})$.* Here we let $(\Pi, k, g) \in T_l^{\text{norm}}(D_{l+1}^{(2)})$ with $l \geq 2$. In Table 4.1, we give the list of k, g, ϑ, k^\vee in the case where $A(\Pi)$ is $D_{l+1}^{(2)}$. We let β denote an element of Π_{lg} and we notice $\{\alpha_0, \alpha_1\} = \Pi_{\text{sh}}$. Each number (r), $i \leq r \leq vi$, in the rightmost column of the table means the Dynkin diagram $\widehat{\Gamma}$ which shall be described in (r) in Fig. 4.6-4.11 after the table is given; (r)* means the one obtained from (r) by replacing α_1, α_i ($2 \leq i \leq l$), α_0 by $\alpha_0, \alpha_{l-i+2}, \alpha_1$ respectively.

$k(\alpha_0)$	$g(\alpha_0)$	$k(\beta)$	$k(\alpha_1)$	$g(\alpha_1)$	$\vartheta(\alpha_0)$	$\vartheta(\alpha_1)$	$k^\vee(\alpha_0)$	$k^\vee(\beta)$	$k^\vee(\alpha_1)$	$\widehat{\Gamma}$
1	\emptyset	1	1	$2\mathbb{Z} + 1$	1	3	2	2	2	(iv)*
1	\emptyset	2	2	$2\mathbb{Z} + 1$	2	3	4	2	2	(v)*
1	\emptyset	1	1	\mathbb{Z}	1	1	1	1	1	(i)
1	\emptyset	2	2	\mathbb{Z}	2	1	2	1	1	(ii)
1	\emptyset	1	1	$2\mathbb{Z}$	1	3	2	2	2	(iv)*
1	\emptyset	2	1	$2\mathbb{Z}$	2	2	2	1	2	(iii)
2	\emptyset	2	1	$2\mathbb{Z}$	1	2	1	1	2	(ii)*
1	\emptyset	2	2	$2\mathbb{Z}$	2	3	4	2	2	(v)*
1	\emptyset	2	1	$4\mathbb{Z}$	2	3	4	2	3	(v)*
2	\emptyset	2	1	$4\mathbb{Z}$	1	3	2	2	3	(iv)*
1	$2\mathbb{Z} + 1$	1	1	$2\mathbb{Z} + 1$	3	3	2	2	2	(vi)
1	$2\mathbb{Z} + 1$	1	1	\mathbb{Z}	3	1	2	2	2	(iv)
1	$2\mathbb{Z} + 1$	1	1	$2\mathbb{Z}$	3	3	2	2	2	(vi)
2	$2\mathbb{Z} + 1$	2	1	$2\mathbb{Z}$	3	2	2	2	4	(v)
2	$2\mathbb{Z} + 1$	2	1	$4\mathbb{Z}$	3	3	2	2	3	(vi)
1	\mathbb{Z}	1	1	\mathbb{Z}	1	1	1	1	1	(i)
1	\mathbb{Z}	1	1	$2\mathbb{Z}$	1	3	2	2	2	(iv)*
2	\mathbb{Z}	2	1	$2\mathbb{Z}$	1	2	1	1	2	(ii)*
2	\mathbb{Z}	2	1	$4\mathbb{Z}$	1	3	2	2	3	(iv)*
1	$2\mathbb{Z}$	1	1	$2\mathbb{Z}$	3	3	2	2	2	(vi)
1	$2\mathbb{Z}$	2	1	$2\mathbb{Z}$	2	2	2	1	2	(iii)
2	$2\mathbb{Z}$	2	1	$2\mathbb{Z}$	3	2	2	2	4	(v)
1	$2\mathbb{Z}$	2	1	$4\mathbb{Z}$	2	3	4	2	3	(v)*
2	$2\mathbb{Z}$	2	1	$4\mathbb{Z}$	3	3	2	2	3	(vi)
1	$4\mathbb{Z}$	2	1	$4\mathbb{Z}$	3	3	3	2	3	(vi)

Table 4.1

The Dynkin diagram $\widehat{\Gamma}$ of $\bar{\mathfrak{G}}(\widehat{A}(\Pi, k, g), \widehat{I}(\Pi, k, g), \widehat{I}^{\text{odd}})$ is given in the following way. In $\widehat{\Gamma}$, the symbol i, x means $(\alpha_i, x) (\in \widehat{I}(\Pi, k, g))$. We call the dot of $\widehat{\Gamma}$ corresponding to (α_i, x) the (α_i, x) -th dot. As for the (α_i, x) -th dot

of $\widehat{\Gamma}$, we mean

$$\begin{cases} (\alpha_i, x) \notin \widehat{\Gamma}^{\text{odd}} \text{ and } \widehat{a}_{(\alpha_i, x), (\alpha_i, x)} = 2 & \text{for (1) of Fig. 4.2,} \\ (\alpha_i, x) \in \widehat{\Gamma}^{\text{odd}} \text{ and } \widehat{a}_{(\alpha_i, x), (\alpha_i, x)} = 2 & \text{for (2) of Fig. 4.2,} \\ (\alpha_i, x) \in \widehat{\Gamma}^{\text{odd}} \text{ and } \widehat{a}_{(\alpha_i, x), (\alpha_i, x)} = 0 & \text{for (3) of Fig. 4.2.} \end{cases}$$

If $\alpha_i \in \Pi_{\text{sh}}$ and $\vartheta(\alpha_i) = 1$, then

$$(4) \text{ of Fig. 4.2 means } \begin{cases} (1) \text{ of Fig. 4.2} & \text{if } g(\alpha_i) = \emptyset, \\ (2) \text{ of Fig. 4.2} & \text{if } g(\alpha_i) = \mathbb{Z}. \end{cases}$$

$$(1) \quad \bigcirc_{i, x} \quad (2) \quad \bullet_{i, x} \quad (3) \quad \otimes_{i, x} \quad (4) \quad \odot_{i, x}$$

Fig. 4.2

If there exist no lines between the (α_i, x) -th dot and the (α_j, y) -th dot, we mean that $\widehat{a}_{(\alpha_i, x), (\alpha_j, y)} = \widehat{a}_{(\alpha_j, y), (\alpha_i, x)} = 0$. Otherwise we mean

$$(\widehat{a}_{(\alpha_i, x), (\alpha_j, y)}, \widehat{a}_{(\alpha_j, y), (\alpha_i, x)}) = \begin{cases} (-1, -1) & \text{for (1) of Fig. 4.3,} \\ (-1, -2) & \text{for (2) of Fig. 4.3,} \\ (-1, -1) & \text{for (3) of Fig. 4.3,} \\ (1, 1) & \text{for (4) of Fig. 4.3,} \\ (2, 2) & \text{for (5) of Fig. 4.3.} \end{cases}$$

$$(1) \quad \bigcirc_{i, x} \text{---} \bigcirc_{j, y} \quad (2) \quad \bigcirc_{i, x} \text{====} \odot_{j, y} \quad (3) \quad \bigcirc_{i, x} \text{---} \otimes_{j, y} \quad (4) \quad \otimes_{i, x} \text{---} \otimes_{j, y} \quad (5) \quad \otimes_{i, x} \text{====} \otimes_{j, y}$$

Fig. 4.3

If $\alpha_i \in \Pi_{\text{sh}}$ and $\vartheta(\alpha_i) = 2$, then

$$(1) \text{ of Fig. 4.4 means } \begin{cases} (2) \text{ of Fig. 4.4} & \text{if } g(\alpha_i) = \emptyset, \\ (3) \text{ of Fig. 4.4} & \text{if } g(\alpha_i) = 2\mathbb{Z}. \end{cases}$$

$$(1) \quad \times_{i, x} \text{====} \times_{i, y} \quad (2) \quad \bigcirc_{i, x} \quad \bigcirc_{i, y} \quad (3) \quad \otimes_{i, x} \text{====} \otimes_{i, y}$$

Fig. 4.4

If $\alpha_i \in \Pi_{\text{sh}}$ and $\vartheta(\alpha_i) = 3$, then

$$(1) \text{ of Fig. 4.5 means } \begin{cases} (2) \text{ of Fig. 4.5} & \text{if } g(\alpha_i) = 2\mathbb{Z}, \\ (3) \text{ of Fig. 4.5} & \text{if } g(\alpha_i) = 2\mathbb{Z} + 1, \\ (4) \text{ of Fig. 4.5} & \text{if } g(\alpha_i) = 4\mathbb{Z}. \end{cases}$$

$$(1) \quad \times \square \times \quad (2) \quad \otimes \text{---} \otimes \quad (3) \quad \circ \text{---} \circ \quad (4) \quad \otimes \text{---} \otimes \text{---} \circ$$

$$i, 1 \quad i, 2 \quad i, 1 \quad i, 2 \quad i, 1 \quad i, 2 \quad i, 1 \quad i, 3 \quad i, 2$$

Fig. 4.5

Then $\widehat{\Gamma}$ is given in the following way. In the following, we assume $\vartheta(\alpha_0) \geq \vartheta(\alpha_1)$. The cases where $\vartheta(\alpha_0) < \vartheta(\alpha_1)$ can be treated similarly.

(i) If $\vartheta(\alpha_0) = \vartheta(\alpha_1) = 1$, then $\widehat{\Gamma}$ is the one of Fig. 4.6.

$$\bullet \leftarrow \circ \text{---} \circ \text{---} \dots \text{---} \circ \Rightarrow \bullet$$

$$0, 1 \quad l, 1 \quad l-1, 1 \quad 2, 1 \quad 1, 1$$

Fig. 4.6

(ii) If $\vartheta(\alpha_0) = 2$ and $\vartheta(\alpha_1) = 1$, then $\widehat{\Gamma}$ is the one of Fig. 4.7.

$$\begin{array}{c} 0, 2 \\ \times \\ \vdots \\ \times \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \circ \text{---} \circ \text{---} \dots \text{---} \circ \Rightarrow \bullet$$

$$0, 1 \quad l, 1 \quad l-1, 1 \quad 2, 1 \quad 1, 1$$

Fig. 4.7

(iii) If $\vartheta(\alpha_0) = \vartheta(\alpha_1) = 2$, then $\widehat{\Gamma}$ is the one of Fig. 4.8.

$$\begin{array}{c} 0, 2 \\ \times \\ \vdots \\ \times \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \circ \text{---} \circ \text{---} \dots \text{---} \circ \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} 1, 2 \\ \times \\ \vdots \\ \times \end{array}$$

$$0, 1 \quad l, 1 \quad l-1, 1 \quad 2, 1 \quad 1, 1$$

Fig. 4.8

(iv) If $\vartheta(\alpha_0) = 3$ and $\vartheta(\alpha_1) = 1$, then $\widehat{\Gamma}$ is the one of Fig. 4.9.

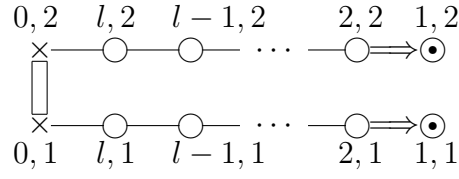


Fig. 4.9

(v) If $\vartheta(\alpha_0) = 3$ and $\vartheta(\alpha_1) = 2$, then $\widehat{\Gamma}$ is the one of Fig. 4.10.

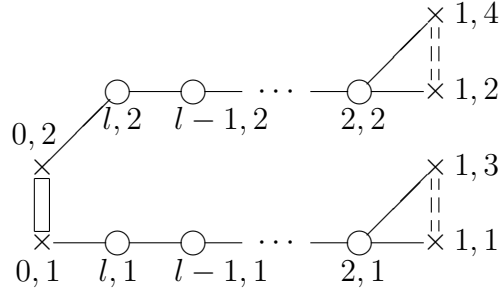


Fig. 4.10

(vi) If $\vartheta(\alpha_0) = \vartheta(\alpha_1) = 3$, then $\widehat{\Gamma}$ is the one of Fig. 4.11.

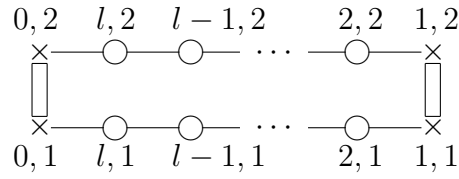


Fig. 4.11

Example 5.1. Assume $l = 3$. Let $(\Pi = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}, k, g) \in T_l^{\text{norm}}(D_{l+1}^{(2)})$.

(i) If $k(\alpha_0) = k(\alpha_1) = k(\alpha_2) = k(\alpha_3) = 1$, $g(\alpha_0) = g(\alpha_2) = g(\alpha_3) = \emptyset$ and $g(\alpha_1) = \mathbb{Z}$, then $\widehat{\Gamma}$ is the one of (i) of Fig. 4.12.

(ii) If $k(\alpha_0) = 1$, $k(\alpha_1) = k(\alpha_2) = k(\alpha_3) = 2$, $g(\alpha_0) = g(\alpha_2) = g(\alpha_3) = \emptyset$, and $g(\alpha_1) = \mathbb{Z}$, then $\widehat{\Gamma}$ is the one of (ii) of Fig. 4.12.

(iii) If $k(\alpha_0) = k(\alpha_1) = 1$, $k(\alpha_2) = k(\alpha_3) = 2$, $g(\alpha_0) = g(\alpha_2) = g(\alpha_3) = \emptyset$, and $g(\alpha_1) = 2\mathbb{Z}$, then $\widehat{\Gamma}$ is the one of (iii) of Fig. 4.12.

(iv)* If $k(\alpha_0) = k(\alpha_1) = k(\alpha_2) = k(\alpha_3) = 1$, $g(\alpha_0) = g(\alpha_2) = g(\alpha_3) = \emptyset$, and $g(\alpha_1) = 2\mathbb{Z}$, then $\widehat{\Gamma}$ is the one of (iv)* of Fig. 4.12.

(v-1) If $k(\alpha_0) = 1$, $k(\alpha_1) = k(\alpha_2) = k(\alpha_3) = 2$, $g(\alpha_0) = g(\alpha_2) = g(\alpha_3) = \emptyset$, and $g(\alpha_1) = 2\mathbb{Z}$, then $\widehat{\Gamma}$ is the one of (v-1) of Fig. 4.12.

(v-2) If $k(\alpha_0) = k(\alpha_1) = 1$, $k(\alpha_2) = k(\alpha_3) = 2$, $g(\alpha_0) = 2\mathbb{Z}$, $g(\alpha_2) = g(\alpha_3) = \emptyset$, and $g(\alpha_1) = 4\mathbb{Z}$, then $\widehat{\Gamma}$ is the one of (v-2) of Fig. 4.12.

(vi) If $k(\alpha_0) = k(\alpha_2) = k(\alpha_3) = 2$, $k(\alpha_1) = 1$, $g(\alpha_0) = 2\mathbb{Z} + 1$, $g(\alpha_2) = g(\alpha_3) = \emptyset$, and $g(\alpha_1) = 4\mathbb{Z}$, then $\widehat{\Gamma}$ is the one of (vi) of Fig. 4.12.

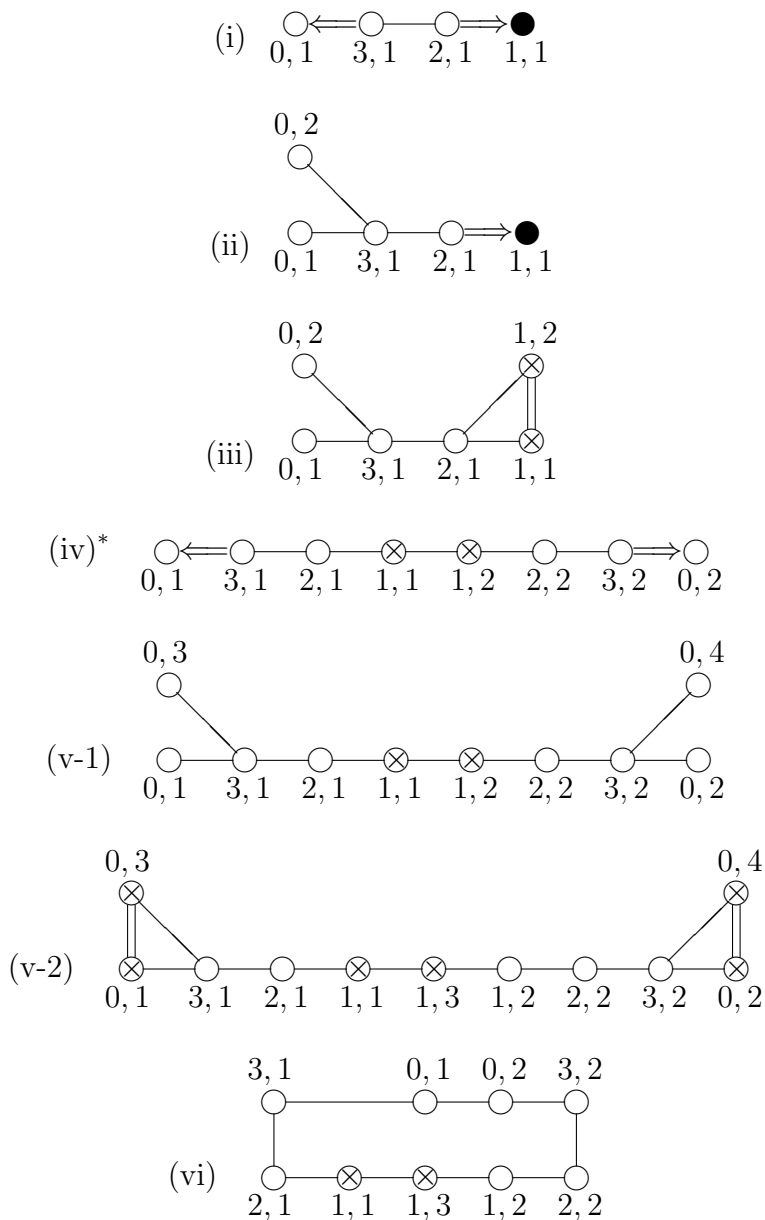


Fig. 4.12

Lemma 5.5. *Let $l \geq 2$ and $(\Pi, k, g) \in T_l^{\text{norm}}(D_{l+1}^{(2)})$ (cf. (5.5)). See (3.26) and (5.8). Let $\zeta = \exp(\frac{\pi\sqrt{-1}}{4}) = \frac{1+\sqrt{-1}}{\sqrt{2}}$. Let $\bar{E}_{-(\alpha,x)} = \bar{F}_{(\alpha,x)}(\in \widehat{\mathfrak{L}})$ (notice $\bar{F}_{(\alpha,x)} \in \bar{\mathfrak{G}}(\widehat{A}, \widehat{I}, \widehat{I}^{\text{odd}}) \subset \widehat{\mathfrak{L}}$). Then there exists a unique homomorphism $\psi_1 : \mathfrak{g}(\Pi, k, g) \rightarrow \widehat{\mathfrak{L}}(\Pi, k, g)$ satisfying the following:*

$$\psi_1(E_{\pm\alpha}) = \begin{cases} \bar{E}_{\pm(\alpha,1)} & \text{if } \alpha \in \Pi_{\text{lg}} \text{ and } k^\vee(\alpha) = 1, \\ \bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)} & \text{if } \alpha \in \Pi_{\text{lg}} \text{ and } k^\vee(\alpha) = 2, \\ \bar{E}_{\pm(\alpha,1)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1 \text{ and } k^\vee(\alpha) = 1, \\ \bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1 \text{ and } k^\vee(\alpha) = 2, \\ \bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 2, \\ \bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)} + \bar{E}_{\pm(\alpha,3)} + \bar{E}_{\pm(\alpha,4)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 4, \\ \sqrt{2}(\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 3 \text{ and } g(\alpha) \in \{2\mathbb{Z} + 1, 2\mathbb{Z}\} \\ & \text{(this implies } k^\vee(\alpha) = 2\text{),} \\ \sqrt{2}(\bar{E}_{\pm(\alpha,1)} \pm [\bar{E}_{\pm(\alpha,3)}, \bar{E}_{\pm(\alpha,2)}]) & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 3 \text{ and } g(\alpha) = 4\mathbb{Z} \\ & \text{(this implies } k^\vee(\alpha) = 3\text{),} \end{cases}$$

$$\psi_1(E_{\pm\alpha^*}) = \begin{cases} \bar{E}_{\pm(\alpha,1)} \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{lg}} \text{ and } k^\vee(\alpha) = 1, \\ \pm\sqrt{-1}(-\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{lg}} \text{ and } k^\vee(\alpha) = 2, \\ \bar{E}_{\pm(\alpha,1)} \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1, g(\alpha) = \emptyset \text{ and } k^\vee(\alpha) = 1, \\ \pm\sqrt{-1}(-\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) \otimes t^{\pm k(\alpha)} & \\ & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1, g(\alpha) = \emptyset \text{ and } k^\vee(\alpha) = 2, \\ \pm\frac{1}{4}[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,1)}] \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1, g(\alpha) = \mathbb{Z} \text{ and } k^\vee(\alpha) = 1, \\ \frac{\sqrt{-1}}{4}(-[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,1)}] + [\bar{E}_{\pm(\alpha,2)}, \bar{E}_{\pm(\alpha,2)}]) \otimes t^{\pm k(\alpha)} & \\ & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1, g(\alpha) = \mathbb{Z} \text{ and } k^\vee(\alpha) = 2, \\ \pm\sqrt{-1}(-\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) \otimes t^{\pm 1} & \\ & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 2 \text{ (this implies } k(\alpha) = 1\text{),} \\ (\zeta^{\mp 3}\bar{E}_{\pm(\alpha,1)} + \zeta^{\mp 1}\bar{E}_{\pm(\alpha,2)} + \zeta^{\pm 1}\bar{E}_{\pm(\alpha,3)} + \zeta^{\pm 3}\bar{E}_{\pm(\alpha,4)}) \otimes t^{\pm 1} & \\ & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 4 \text{ (this implies } k(\alpha) = 1\text{),} \\ \pm\sqrt{-2}(-\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) \otimes t^{\pm k(\alpha)} & \\ & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 3 \text{ and } g(\alpha) = 2\mathbb{Z} \text{ (this implies } k^\vee(\alpha) = 2\text{),} \\ \sqrt{-1}[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,2)}] \otimes t^{\pm k(\alpha)} & \\ & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 3 \text{ and } g(\alpha) = 2\mathbb{Z} + 1 \text{ (this implies } k^\vee(\alpha) = 2\text{),} \\ \sqrt{2}(\bar{E}_{\pm(\alpha,2)} + \sqrt{-1}[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,3)}]) \otimes t^{\pm 1} & \\ & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 3 \text{ and } g(\alpha) = 4\mathbb{Z} \text{ (this implies } k^\vee(\alpha) = 3 \text{ and } k(\alpha) = 1\text{),} \end{cases}$$

$$\psi_1(h_{\alpha^\vee}) = \begin{cases} \sum_{x=1}^{k^\vee(\alpha)} \bar{h}_{(\alpha,x)} & \text{if } \alpha \in \Pi_{\text{lg}}, \\ \sum_{x=1}^{k^\vee(\alpha)} \bar{h}_{(\alpha,x)} & \text{if } \alpha \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha) \in \{1, 2\}, \\ 2 \sum_{x=1}^{k^\vee(\alpha)} \bar{h}_{(\alpha,x)} & \text{if } \alpha \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha) = 3, \end{cases}$$

and $\psi_1(h_a) = \frac{k^\vee(\beta)(\beta,\beta)}{(\bar{\alpha}(\beta,1), \bar{\alpha}(\beta,1))} \bar{v}$ ($\beta \in \Pi_{\text{lg}}$), $\psi_1(h_{\Lambda_a}) = \bar{w}$,

$$\psi_1(h_{\Lambda_\delta}) = \begin{cases} \bar{t}_{(\alpha_0,1)} + \bar{t}_{(\alpha_0,2)} & \text{if } g(\alpha_0) = 4\mathbb{Z}, \\ \sum_{x=1}^{k^\vee(\alpha_0)} \bar{t}_{(\alpha_0,x)} & \text{otherwise.} \end{cases}$$

In particular, $\dim \mathfrak{h}(\Pi, k, g) = l + 4$.

This can be checked directly by using Lemma 5.3. We can also use (5.1)-(5.2) and Remark 8.1.

Remark 5.3. The ideas of k^\vee , \widehat{A} , \widehat{I}^{odd} and ψ_1 have been inspired by explicit constructions of the affine Lie superalgebras in [16]. In fact, for $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ and $(\alpha, \beta) \neq 0$, $\psi_1(\mathfrak{g}^{\{\alpha, \beta\}})$ gives an explicit construction of the rank-two affine Lie superalgebra $\mathfrak{g}^{\{\alpha, \beta\}}$ similar to that in [16, Theorem 6.11 (and Table 4)].

5.6 From $A_{2l}^{(2)}$ to $D_{l+1}^{(2)}$

Let $l \geq 2$ and $(\Pi = \{\alpha_0, \dots, \alpha_l\}, k, g) \in T_l^{\text{norm}}(A_{2l}^{(2)})$ (see also Subsection 5.4). Notice that $\Pi_{\text{sh}} = \{\alpha_1\}$, $\Pi_{\text{lg}} = \{\alpha_2, \dots, \alpha_l\}$ and $\Pi_{\text{ex}} = \{\alpha_0\}$. We define $(\Pi' = \{\alpha'_0, \dots, \alpha'_l\}, k', g') \in T_l^{\text{norm}}(D_{l+1}^{(2)})$ to be as follows. Let

$$\begin{cases} \beta'_i = \alpha'_i \ (0 \leq i \leq l) & \text{if } g(\alpha_1) \in \{2\mathbb{Z}, 4\mathbb{Z}\} \\ \beta'_0 = \alpha'_1, \beta'_1 = \alpha'_0, \beta'_i = \alpha'_{l-i+2} \ (2 \leq i \leq l) & \text{if } g(\alpha_1) \in \{2\mathbb{Z} + 1, \mathbb{Z}\} \end{cases}$$

and $m_0 = 1 + \delta_{k(\alpha_0), 1}$. Then we let

$$k'(\beta'_j) = \frac{k(\alpha_j)m_0}{1 + \delta_{j,0}}$$

($0 \leq j \leq l$) and

$$g'(\beta'_r) = \begin{cases} 2\mathbb{Z} & \text{if } r = 0, \\ g(\alpha_r) & \text{if } 1 \leq r \leq l. \end{cases}$$

Denote a for (Π', k', g') by a' and let $\delta' = \delta(\Pi') (= \sum_{i=0}^l \beta'_i)$ (cf. (3.18)). Assume $(\beta'_1, \beta'_1) = (\alpha_1, \alpha_1)$. Let

$$\Lambda_{\delta'} = \begin{cases} \Lambda_{\delta'} & \text{if } \beta'_0 = \alpha'_0, \\ \Lambda_{\delta'} - \frac{l}{2(\alpha'_1, \alpha'_1)} \delta' + \frac{1}{(\alpha'_1, \alpha'_1)} \sum_{i=1}^l (l - i + 1) \alpha'_i & \text{if } \beta'_0 = \alpha'_1. \end{cases}$$

Lemma 5.6. *Keep the notation as above. See (3.26). Then there exists a homomorphism $\psi_2 : \mathfrak{g}(\Pi, k, g) \rightarrow \mathfrak{g}(\Pi', k', g')$ such that $\psi_2(h_{\alpha_0^\vee}) = \frac{1}{2}h_{(\beta'_0)^\vee}$, $\psi_2(E_{\pm\alpha_0}) = \pm\frac{1}{4}[E_{\pm\beta'_0}, E_{\pm\beta'_0}]$, $\psi_2(E_{\pm\alpha_0^*}) = \pm\frac{1}{4}[E_{\pm(\beta'_0)^*}, E_{\pm(\beta'_0)^*}]$, $\psi_2(E_{\pm\alpha_i}) = E_{\pm\beta'_i}$, $\psi_2(E_{\pm\alpha_i^*}) = E_{\pm(\beta'_i)^*}$, $\psi_2(h_{\alpha_i^\vee}) = h_{(\beta'_i)^\vee}$, ($1 \leq i \leq l$), $\psi_2(h_a) = m_0 h_{a'}$, $\psi_2(h_{\Lambda_a}) = \frac{1}{m_0} h_{\Lambda_{a'}}$ and $\psi_2(h_{\Lambda_\delta}) = \frac{1}{2} h_{\Lambda_{\delta'}}$. Moreover, $\dim \mathfrak{h}(\Pi, k, g) = l + 4$.*

Proof. Using (5.1)-(5.2) and Remark 8.1, we can easily check that ψ_2 is a homomorphism. We see $\dim \mathfrak{h}(\Pi, k, g) = l + 4$ by Lemma 5.5 since $\psi_2(\mathfrak{h}(\Pi, k, g)) = \mathfrak{h}(\Pi', k', g')$. \square

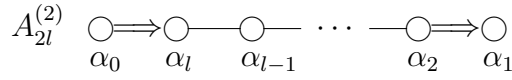


Fig. 4.13

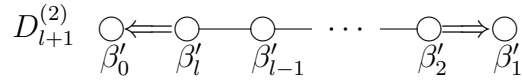


Fig. 4.14

In the following table, $\beta \in \Pi_{\text{lg}}$ and $\beta' \in \Pi'_{\text{lg}}$.

$k(\alpha_0)$	$k(\beta)$	$k(\alpha_1)$	$g(\alpha_1)$	$k'(\beta'_0)$	$g'(\beta'_0)$	$k'(\beta')$	$k'(\beta'_1)$	$g'(\beta'_1)$
1	1	1	$2\mathbb{Z} + 1$	1	$2\mathbb{Z}$	2	2	$2\mathbb{Z} + 1$
2	1	1	$2\mathbb{Z} + 1$	1	$2\mathbb{Z}$	1	1	$2\mathbb{Z} + 1$
1	1	1	\mathbb{Z}	1	$2\mathbb{Z}$	2	2	\mathbb{Z}
2	1	1	\mathbb{Z}	1	$2\mathbb{Z}$	1	1	\mathbb{Z}
1	1	1	$2\mathbb{Z}$	1	$2\mathbb{Z}$	2	2	$2\mathbb{Z}$
2	1	1	$2\mathbb{Z}$	1	$2\mathbb{Z}$	1	1	$2\mathbb{Z}$
2	2	1	$2\mathbb{Z}$	1	$2\mathbb{Z}$	2	1	$2\mathbb{Z}$
4	2	1	$2\mathbb{Z}$	2	$2\mathbb{Z}$	2	1	$2\mathbb{Z}$
2	2	1	$4\mathbb{Z}$	1	$2\mathbb{Z}$	2	1	$4\mathbb{Z}$
4	2	1	$4\mathbb{Z}$	2	$2\mathbb{Z}$	2	1	$4\mathbb{Z}$

Table 4.2.

5.7 From $B_l^{(1)}$ to $D_{l+1}^{(2)}$

Let $l \geq 3$ and $(\Pi, k, g) \in T_l^{\text{norm}}(B_l^{(1)})$ (see also Subsection 5.4). Notice that $\Pi_{\text{sh}} = \{\alpha_1\}$ and $\Pi_{\text{lg}} = \{\alpha_0, \alpha_2, \dots, \alpha_l\}$. We let $(\Pi' = \{\alpha'_0, \dots, \alpha'_l\}, k', g') \in T_l^{\text{norm}}(D_{l+1}^{(2)})$ be such that $k'(\alpha'_i) = k(\alpha_i)$ and $g'(\alpha'_i) = g(\alpha_i)$ ($0 \leq i \leq l$). Denote a for (Π', k', g') by a' and let $\delta' = \delta(\Pi')$ (cf. (3.18)). Assume $(\alpha'_1, \alpha'_1) = (\alpha_1, \alpha_1)$.

Lemma 5.7. *Keep the notation as above. See (3.26). Then there exists a homomorphism $\psi_3 : \mathfrak{g}(\Pi, k, g) \rightarrow \mathfrak{g}(\Pi', k', g')$ such that $\psi_3(h_{\alpha_0^\vee}) = h_{(\alpha'_0)^\vee} + h_{(\alpha'_1)^\vee}$, $\psi_3(E_{\pm\alpha_0}) = \frac{1}{2}(\text{ad}E_{\pm\alpha'_0})^2(E_{\pm\alpha'_1})$, $\psi_3(E_{\pm\alpha_0^*}) = \frac{1}{2}(\text{ad}E_{\pm\alpha'_0})^2(E_{\pm(\alpha'_1)^*})$, $\psi_3(E_{\pm\alpha_i}) = E_{\pm\alpha'_i}$, $\psi_3(E_{\pm\alpha_i^*}) = E_{\pm(\alpha'_i)^*}$, $\psi_3(h_{\alpha_i^\vee}) = h_{(\alpha'_i)^\vee}$ ($1 \leq i \leq l$), $\psi_3(h_a) = h_{a'}$, $\psi_3(h_{\Lambda_a}) = h_{\Lambda_{a'}}$ and $\psi_3(h_{\Lambda_\delta}) = \frac{1}{2}h_{\Lambda_{\delta'}}$. Moreover, $\dim \mathfrak{h}(\Pi, k, g) = l + 4$.*

Proof. Let $\gamma' \in \{\alpha'_i, (\alpha'_i)^*\}$. By (4.10), $[n_{\alpha'_0}(E_{\gamma'}), n_{\alpha'_0}(E_{-\gamma'})] = n_{\alpha'_0}(h_{(\gamma')^\vee}) = h_{(\alpha'_0)^\vee} + h_{(\gamma')^\vee}$. By (4.13), we have $n_{\alpha'_0}(E_{\pm\gamma'}) = \frac{1}{2}(\text{ad}E_{\pm\alpha'_0})^2(E_{\pm\gamma'})$. Hence we have (SR4) for $\psi_3(E_{\pm\gamma'})$, where $\gamma \in \{\alpha_0, \alpha_0^*\}$. Since $[E_{\pm\alpha'_0}, E_{\pm(\alpha'_1)^*}] = [E_{\pm(\alpha'_0)^*}, E_{\pm\alpha'_1}]$ (by (SR6,7)), we can easily see $[\psi_3(E_{\pm\alpha_0^*}), \psi_3(E_{\mp\alpha_1})] = 0$. By (4.12), we have $[\psi_3(E_{\pm\alpha_0}), \psi_3(E_{\mp\alpha_1^*})] = n_{\alpha'_0}([\psi_3(E_{\pm\alpha_1}), \psi_3(E_{\mp\alpha_0^*})]) = 0$. Then, using (5.1)-(5.2) and Remark 8.1, we can easily check that ψ_3 is a homomorphism. We see $\dim \mathfrak{h}(\Pi, k, g) = l + 4$ by Lemma 5.5 since $\psi_3(\mathfrak{h}(\Pi, k, g)) = \mathfrak{h}(\Pi', k', g')$. \square

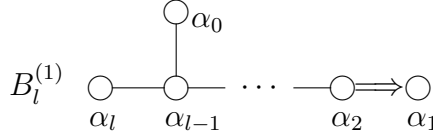


Fig. 4.15

In the following table, $\beta \in \Pi_{\text{lg}}$.

$k(\beta)$	$k(\alpha_1)$	$g(\alpha_1)$
1	1	$2\mathbb{Z} + 1$
1	1	\mathbb{Z}
1	1	$2\mathbb{Z}$
2	1	$2\mathbb{Z}$
2	1	$4\mathbb{Z}$

Table 4.3.

5.8 From $C_2^{(1)}$ to $B_3^{(1)}$ or $F^{(1)}(4)$

In this subsection, we assume $(\Pi, k, g) \in T_2^{\text{norm}}(C_2^{(1)})$ (see also Subsection 5.4). Then $l \geq 2$, $\Pi_{\text{sh}} = \{\alpha_1\}$ and $\Pi_{\text{lg}} = \{\alpha_0, \alpha_2\}$.

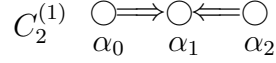


Fig. 4.16

$k(\alpha_0)$	$k(\alpha_1)$	$g(\alpha_1)$	$k(\alpha_2)$
1	1	$2\mathbb{Z} + 1$	1
1	1	\mathbb{Z}	1
1	1	$2\mathbb{Z}$	1
2	1	$2\mathbb{Z}$	1
2	1	$2\mathbb{Z}$	2
2	1	$4\mathbb{Z}$	2

Table 4.4

Assume $k(\alpha_0) = k(\alpha_2)$. Let $(\Pi' = \{\alpha'_0, \alpha'_1, \alpha'_2, \alpha'_3\}, k', g') \in T_3^{\text{norm}}(B_3^{(1)})$ be such that $k'(\alpha'_0) = k'(\alpha'_2) = k'(\alpha'_3) = k(\alpha_0)$, $k'(\alpha'_1) = 1$, $g'(\alpha'_0) = g'(\alpha'_2) = g'(\alpha'_3) = \emptyset$ and $g'(\alpha'_1) = g(\alpha_1)$. Denote a for (Π', k', g') by a' and let $\delta' = \delta(\Pi')$ (cf. (3.18)). Assume $(\alpha'_1, \alpha'_1) = (\alpha_1, \alpha_1)$.

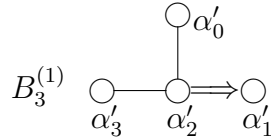


Fig. 4.17

Lemma 5.8. *Keep the notation as above. See (3.26). Then there exists a homomorphism $\psi_4 : \mathfrak{g}(\Pi, k, g) \rightarrow \mathfrak{g}(\Pi', k', g')$ such that $\psi_4(E_{\pm\alpha_1}) = E_{\pm\alpha'_1}$, $\psi_4(E_{\pm\alpha'_1}) = E_{\pm(\alpha'_1)^*}$, $\psi_4(E_{\pm\alpha_0}) = \pm[E_{\pm\alpha'_0}, E_{\pm\alpha'_2}]$, $\psi_4(E_{\pm\alpha_2}) = \pm[E_{\pm\alpha'_3}, E_{\pm\alpha'_2}]$, $\psi_4(E_{\pm\alpha'_0}) = \pm[E_{\pm\alpha'_0}, E_{\pm(\alpha'_2)^*}]$, $\psi_4(E_{\pm\alpha'_2}) = \pm[E_{\pm\alpha'_3}, E_{\pm(\alpha'_2)^*}]$, $\psi_4(h_{\alpha'_0}) = h_{(\alpha'_0)^\vee} + h_{(\alpha'_2)^\vee}$, $\psi_4(h_{\alpha'_2}) = h_{(\alpha'_3)^\vee} + h_{(\alpha'_2)^\vee}$, $\psi_4(h_{\alpha'_1}) = h_{(\alpha'_1)^\vee}$, $\psi_4(h_a) = h_{a'}$, $\psi_4(h_{\Lambda_a}) = h_{\Lambda_{a'}}$ and $\psi_4(h_{\Lambda_\delta}) = h_{\Lambda_{\delta'}}$. Moreover, $\dim \mathfrak{h}(\Pi, k, g) = 6$.*

Proof. We can prove this in a way similar to that in Proof of Lemma 5.7. We can use the formula $n_{\alpha'_i}(E_{\pm\alpha'_2}) = \pm[E_{\pm\alpha'_i}, E_{\pm\alpha'_2}]$ and $n_{\alpha'_i}(E_{\pm(\alpha'_2)^*}) = \pm[E_{\pm\alpha'_i}, E_{\pm(\alpha'_2)^*}]$ ($i \in \{0, 3\}$) (see also (4.10) and (4.13)). We can also use (5.1)-(5.2) and Remark 8.1. Using ψ_4 , by Lemma 5.7, we see $\dim \mathfrak{h}(\Pi, k, g) = 6$. \square

Assume $k(\alpha_0) \neq k(\alpha_2)$. Then we see that such (Π, k, g) exists uniquely and is such that $k(\alpha_1) = k(\alpha_2) = 1$, $k(\alpha_0) = 2$, $g(\alpha_1) = 2\mathbb{Z}$ and $g(\alpha_0) = g(\alpha_2) = \emptyset$.

Let $\bar{I} = \{0, 1, 2, 3, 4\}$ and $\bar{I}^{\text{odd}} = \{1, 2\}$. Define a 5×5 matrix \bar{A} by

$$\bar{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & -1 \\ 0 & 2 & 0 & -2 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix}.$$

The Dynkin diagram of $\bar{\mathfrak{G}}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$ is given in Fig. 4.18. We notice that $\bar{\mathfrak{G}}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$ is the affine Lie superalgebra $F(4)^{(1)}$ (cf. [16, Table 3(b)]).

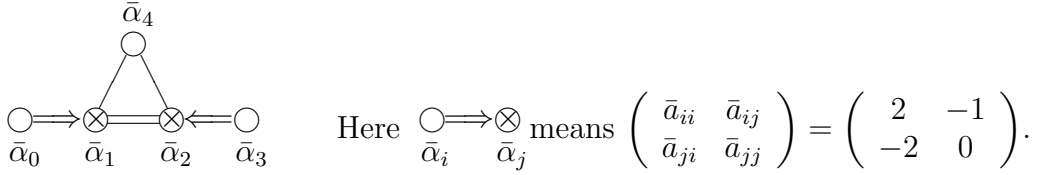


Fig. 4.18

Lemma 5.9. *Keep the notation as above. See (3.26) and (5.3). Then there exists a homomorphism $\psi_5 : \mathfrak{g}(\Pi, k, g) \rightarrow \mathfrak{L}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$ such that $\psi_5(E_{\alpha_2}) = \bar{E}_0 + \bar{E}_3$, $\psi_5(E_{-\alpha_2}) = \bar{F}_0 + \bar{F}_3$, $\psi_5(E_{\alpha_1}) = \bar{E}_1 + \bar{E}_2$, $\psi_5(E_{-\alpha_1}) = \bar{F}_1 + \bar{F}_2$, $\psi_5(E_{\alpha_0}) = \bar{E}_4$, $\psi_5(E_{-\alpha_0}) = \bar{F}_4$, $\psi_5(E_{\alpha_2^*}) = -\sqrt{-1}(\bar{E}_0 - \bar{E}_3) \otimes t$, $\psi_5(E_{-\alpha_2^*}) = \sqrt{-1}(\bar{F}_0 - \bar{F}_3) \otimes t^{-1}$, $\psi_5(E_{\alpha_1^*}) = -\sqrt{-1}(\bar{E}_1 - \bar{E}_2) \otimes t$, $\psi_5(E_{-\alpha_1^*}) = \sqrt{-1}(\bar{F}_1 - \bar{F}_2) \otimes t^{-1}$, $\psi_5(E_{\alpha_0^*}) = \bar{E}_4 \otimes t^2$, $\psi_5(E_{-\alpha_0^*}) = \bar{F}_4 \otimes t^{-2}$, $\psi_5(h_{\alpha_2^\vee}) = \bar{h}_0 + \bar{h}_3$, $\psi_5(h_{\alpha_0^\vee}) = \bar{h}_4$, $\psi_5(h_{\alpha_1^\vee}) = \bar{h}_1 + \bar{h}_2$, $\psi_5(h_{\Lambda_a}) = \bar{w}$, $\psi_5(h_{\Lambda_s}) = \bar{t}_4$ and $\psi_5(h_a) = \frac{(\alpha_0, \alpha_0)}{(\bar{\alpha}_4, \bar{\alpha}_4)} \bar{v}$. Moreover, $\dim \mathfrak{h}(\Pi, k, g) = 6$.*

This can be proved directly by using Lemma 5.3. We can also use (5.1)-(5.2) and Remark 8.1.

5.9 Proof of Lemma 4.1

Proof of Lemma 4.1. The lemma follows immediately from Lemmas 5.5-5.9. \square

6 Invariant form (Proof of Theorem 1.1 (2))

6.1 Normalization of an invariant form

Lemma 6.1. *Let $l \geq 2$ and $(\Pi, k, g) \in T_l^{\text{norm}}$ (see (3.28)). Let $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$ be of (3.26). Let \mathcal{B}_+ and $\mathcal{E} = \mathcal{E}(\Pi, k, g)$ be of (3.24). Assume that there exists a super-symmetric invariant form $(,)' : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ such that*

$$(6.1) \quad (h_\mu, h_\mu)' \neq 0 \quad \text{for some } \mu \in \mathcal{B}_+.$$

Then there exists a super-symmetric invariant form $(,) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ such that

$$(6.2) \quad (h_\sigma, h_\tau) = (\sigma, \tau) \quad \text{for } \sigma, \tau \in \mathcal{E}.$$

and

$$(6.3) \quad \ker(,) \subset \bigoplus_{(m,r) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \mathfrak{g}^{m\delta+ra},$$

where $\delta = \delta(\Pi)$ (see (3.18)).

Proof. (*Strategy.* We first show that there exists $(,)$ satisfying (6.2) for $\sigma \in \mathcal{B}_+$ and $\tau \in \mathcal{E}$ (see Step 1). Then we modify it so that it satisfies (6.2) (see Step 2). Finally we show (6.3) by using Lemma 4.4 (3) and the fact $(E_\mu, E_{-\mu}) = \frac{2}{(\mu, \mu)} \neq 0$ ($\mu \in \mathcal{B}_+$) (see Step 3).)

We proceed in steps.

Step 1. There exists $x \in \mathbb{C} \setminus \{0\}$ such that

$$(6.4) \quad (h_\sigma, h_\mu)' = x(\sigma, \mu) \quad \text{for } \sigma \in \mathcal{E} \text{ and } \mu \in \mathcal{B}_+.$$

For $\mu \in \mathcal{B}_+$, we have

$$\begin{aligned}
(6.5) \quad (E_\mu, E_{-\mu})' &= \frac{1}{(\mu, \mu)} ([h_\mu, E_\mu], E_{-\mu})' \quad (\text{by (SR3)}) \\
&= \frac{1}{(\mu, \mu)} (h_\mu, [E_\mu, E_{-\mu}])' \\
&= \frac{2}{(\mu, \mu)^2} (h_\mu, h_\mu)' \quad (\text{by (SR4)}).
\end{aligned}$$

Then, for $\sigma \in \mathcal{E}$ and $\mu \in \mathcal{B}_+$, we have

$$\begin{aligned}
(6.6) \quad (h_\sigma, h_\mu)' &= (h_\sigma, \frac{(\mu, \mu)}{2} [E_\mu, E_{-\mu}])' \quad (\text{by (SR4)}) \\
&= \frac{(\mu, \mu)}{2} ([h_\sigma, E_\mu], E_{-\mu})' \\
&= \frac{(\mu, \mu)(\sigma, \mu)}{2} (E_\mu, E_{-\mu})' \quad (\text{by (SR3)}) \\
&= (\sigma, \mu) \frac{(h_\mu, h_\mu)'}{(\mu, \mu)} \quad (\text{by (6.5)}).
\end{aligned}$$

Let $\mu \in \mathcal{B}_+$ be of (6.1) and let $x = \frac{(h_\mu, h_\mu)'}{(\mu, \mu)}$, so $x \neq 0$. By (6.6), $x = \frac{(h_\nu, h_\nu)'}{(\nu, \nu)}$ for all $\nu \in \mathcal{B}_+$ since there exist $\nu_i \in \mathcal{B}_+$ ($1 \leq i \leq r$) such that $(\nu_i, \nu_{i+1})' \neq 0$, $\nu_1 = \nu$ and $\nu_r = \mu$. Hence, by (6.6), we have (6.4), as desired.

Step 2. $(,)$ satisfying (6.2) exists. Let $(,)'' = \frac{1}{x} (,)'$ (see Step 1 for $(,)'$). Let

$$\begin{cases} h'_{\Lambda_\delta} = h_{\Lambda_\delta} - \frac{1}{2}((h_{\Lambda_\delta}, h_{\Lambda_\delta})'' h_\delta + (h_{\Lambda_\delta}, h_{\Lambda_a})'' h_a), \\ h'_{\Lambda_a} = h_{\Lambda_a} - \frac{1}{2}((h_{\Lambda_\delta}, h_{\Lambda_a})'' h_\delta + (h_{\Lambda_a}, h_{\Lambda_a})'' h_a). \end{cases}$$

By (6.4), we have

$$(6.7) \quad \begin{cases} (h'_{\Lambda_\delta}, h'_{\Lambda_\delta})'' = (h'_{\Lambda_\delta}, h'_{\Lambda_a})'' = (h'_{\Lambda_a}, h'_{\Lambda_a})'' = 0, \\ (h'_{\Lambda_\delta}, h_\delta)'' = (h'_{\Lambda_a}, h_a)'' = 1, \quad (h'_{\Lambda_\delta}, h_a)'' = (h'_{\Lambda_a}, h_\delta)'' = 0, \\ (h'_{\Lambda_\delta}, h_\alpha)'' = (h'_{\Lambda_a}, h_\alpha)'' = 0 \quad \text{for } \alpha \in \Pi \setminus \{\alpha_0\}. \end{cases}$$

Define $\phi \in \text{Aut}(\mathfrak{g})$ by $\phi(E_{\pm\mu}) = E_{\pm\mu}$, $\phi(h_\mu) = h_\mu$ ($\mu \in \mathcal{B}_+$), $\phi(h_{\Lambda_\delta}) = h'_{\Lambda_\delta}$ and $\phi(h_{\Lambda_a}) = h'_{\Lambda_a}$. Let $(,) = (,)'' \circ (\phi \times \phi)$. Then, (6.4) and (6.7) implies (6.2), as desired.

Step 3. (6.3) holds. For $\sigma, \tau \in \mathcal{E}$ with $\sigma + \tau \neq 0$, we have

$$(6.8) \quad (\mathfrak{g}_\sigma, \mathfrak{g}_\tau) = \{0\}$$

because $0 = ([h_\lambda, X], Y) + (X, [h_\lambda, Y]) = (\lambda, \sigma + \tau)(X, Y)$ for $X \in \mathfrak{g}_\sigma, Y \in \mathfrak{g}_\tau$ and $\lambda \in \mathcal{E}$. By (6.2) (and (6.5)), we have

$$(6.9) \quad \begin{cases} (E_\mu, E_{-\mu}) = \frac{2}{(\mu, \mu)} \neq 0 & \text{if } \mu \in \mathcal{B}_+, \\ ([E_\mu, E_\mu], [E_{-\mu}, E_{-\mu}]) = -4(E_\mu, E_{-\mu}) \neq 0 & \text{if } \mu \in \mathcal{B}_+ \text{ and } p(\mu) = 1. \end{cases}$$

For $\mu \in \mathcal{B}_+$, we can see that for all $X, Y \in \mathfrak{g}$, $(n_\mu(X), n_\mu(Y)) = (X, Y)$ (see (4.9) for $n_\mu \in \text{Aut}(\mathfrak{g})$). Then, by (4.10), (6.9), Lemma 4.4 (3) and Lemma 4.10 (3), we see that for $\mu \in R(\Pi, k, g)$,

$$(6.10) \quad (\cdot, \cdot)|_{\mathfrak{g}_\mu \times \mathfrak{g}_{-\mu}} \text{ is non-degenerate.}$$

By (6.2), (6.8), (6.10) and (4.73), we have (6.3), as desired. This completes the proof. \square

6.2 Proof of Theorem 1.1 (2)

Proof of Theorem 1.1 (2). (See Subsection 1.8 for strategy.) Recall the maps ψ_i ($1 \leq i \leq 5$) from Lemmas 5.5-5.9. Let $(\Pi, k, g) \in T_l^{\text{norm}}$ (see (3.28)) be such that $R = R(\Pi, k, g)$ is of (1.19) (and (3.15)). (We have proved Theorem 3.2.) Define a super-symmetric invariant form $(\cdot, \cdot)' : \mathfrak{g}(\Pi, k, g) \times \mathfrak{g}(\Pi, k, g) \rightarrow \mathbb{C}$ by

$$(6.11) \quad (\cdot, \cdot)' = \begin{cases} (\cdot, \cdot) \circ (\psi_1 \times \psi_1) & \text{if } (\Pi, k, g) \in T_l^{\text{norm}}(D_{l+1}^{(2)}) \ (l \geq 2), \\ (\cdot, \cdot) \circ ((\psi_1 \circ \psi_2) \times (\psi_1 \circ \psi_2)) & \\ \quad \text{if } (\Pi, k, g) \in T_l^{\text{norm}}(A_{2l}^{(2)}) \ (l \geq 2), \\ (\cdot, \cdot) \circ ((\psi_1 \circ \psi_3) \times (\psi_1 \circ \psi_3)) & \\ \quad \text{if } (\Pi, k, g) \in T_l^{\text{norm}}(B_l^{(1)}) \ (l \geq 3), \\ (\cdot, \cdot) \circ ((\psi_1 \circ \psi_3 \circ \psi_4) \times (\psi_1 \circ \psi_3 \circ \psi_4)) & \\ \quad \text{if } (\Pi, k, g) \in T_l^{\text{norm}}(C_2^{(1)}) \ (l = 2) \text{ and } k(\alpha_0) = k(\alpha_1), \\ (\cdot, \cdot) \circ (\psi_5 \times \psi_5) & \\ \quad \text{if } (\Pi, k, g) \in T_l^{\text{norm}}(C_2^{(1)}) \ (l = 2) \text{ and } k(\alpha_0) > k(\alpha_1), \end{cases}$$

where (\cdot, \cdot) of RHS is the one of (5.4). Then using Lemma 6.1, by case-by-case checking, we can easily see that the statement holds. This completes the proof. \square

7 Universality (Proof of Theorem 1.1 (3))

7.1 Characterizations of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{osp}(2|1)$

We give the following lemma and use it in the next subsection.

Lemma 7.1. *Let $\mathfrak{a} = \mathfrak{a}(0) \oplus \mathfrak{a}(1)$ be a Lie superalgebra (see (1.26) for $\mathfrak{a}(i)$). Assume that \mathfrak{a} satisfies the conditions (i)-(iii) below:*

- (i) *There exists a non-degenerate super-symmetric invariant form $(,) : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathbb{C}$.*
- (ii) *There exist $H \in \mathfrak{a}(0)$ and $r \in \{1, 2\}$ such that $(H, H) \neq 0$ and $\mathfrak{a} = \bigoplus_{m=-r}^r \mathfrak{a}_m$, where $\mathfrak{a}_m = \{X \in \mathfrak{a} \mid [H, X] = 2mX\}$.*
- (iii) *For $-r \leq m \leq r$, $\dim \mathfrak{a}_m = 1$ (notice $\mathfrak{a}_0 = \mathbb{C}H$).*

Then we have the following:

- (1) *If $r = 1$, then $\mathfrak{a} = \mathfrak{a}(0)$ and there exist $E \in \mathfrak{a}_1$ and $F \in \mathfrak{a}_{-1}$ such that $[E, F] = H$ (i.e., $\mathfrak{a} \cong \mathfrak{sl}_2(\mathbb{C})$).*
- (2) *If $r = 2$, then $\mathfrak{a}(0) = \mathfrak{a}_{-2} \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_2$, $\mathfrak{a}(1) = \mathfrak{a}_{-1} \oplus \mathfrak{a}_1$ and there exist $E \in \mathfrak{a}_1$ and $F \in \mathfrak{a}_{-1}$ such that $[E, F] = H$, $\mathfrak{a}_2 = \mathbb{C}[E, E]$ and $\mathfrak{a}_{-2} = \mathbb{C}[F, F]$ (i.e., $\mathfrak{a} \cong \mathfrak{osp}(2|1)$).*

This lemma seems well-known, so we give a proof of this lemma in Section 8.

7.2 Proof of Theorem 1.1 (3)

Proof of Theorem 1.1 (3) (See Subsection 1.8 for strategy.)

We proceed in steps.

Step 1: For each $\mu \in \mathcal{B}_+$, $\mathfrak{g}'_{\pm\mu} \subset \mathfrak{g}'(p(\mu))$ and there exist $E'_{\pm\mu} \in \mathfrak{g}'_{\pm\mu}$ satisfying (SR4) and $\mathbb{C}E'_{\pm\mu} = \mathfrak{g}'_{\pm\mu}$. (See (1.2) for $p(\mu)$ and see (1.26) for $\mathfrak{g}'(i)$.) Let $(,)$ be the super-symmetric invariant form of (3-iii). As in (6.8), we have

$$(7.1) \quad (\mathfrak{g}'_{\sigma}, \mathfrak{g}'_{\tau}) = \{0\} \quad \text{if } \sigma + \tau \neq 0.$$

By (3-iii) (see also (1.34)) and (7.1), we see that

$$(7.2) \quad (,)_{|\mathfrak{g}'_{\lambda} \times \mathfrak{g}'_{-\lambda}} \text{ is non-degenerate for } \lambda \in R \cup \{0\}$$

(see also (3-i)-(3-ii) and notice $\mathfrak{g}'_0 = \mathfrak{h}$). Let $\nu \in R$. By (7.2) and by (3-ii) (see also (1.32)), there exist $X'_{\pm\nu} \in \mathfrak{g}'_{\pm\nu}$ such that

$$(7.3) \quad \mathfrak{g}'_{\pm\nu} = \mathbb{C}X'_{\pm\nu} \quad \text{and} \quad (X'_\nu, X'_{-\nu}) = \frac{2}{(\nu, \nu)}.$$

By (7.3), for $\sigma \in \mathcal{E}$, we have

$$(h_\sigma, [X'_\nu, X'_{-\nu}]) = ([h_\sigma, X'_\nu], X'_{-\nu}) = (\sigma, \nu)(X'_\nu, X'_{-\nu}) = \frac{2(\sigma, \nu)}{(\nu, \nu)} = (h_\sigma, h_{\nu^\vee}).$$

Hence, by (7.2) for $\lambda = 0$, we have

$$(7.4) \quad [X'_\nu, X'_{-\nu}] = h_{\nu^\vee}.$$

Let $\mu \in \mathcal{B}_+$ and set

$$\mathfrak{g}'\langle\mu\rangle = \mathbb{C}h_{\mu^\vee} \bigoplus \left(\bigoplus_{m=1}^{p(\mu)+1} (\mathfrak{g}'_{m\mu} \oplus \mathfrak{g}'_{-m\mu}) \right).$$

By (3-ii) (see also (1.3)) and (7.3)-(7.4), $\mathfrak{g}'\langle\mu\rangle$ is a sub-Lie superalgebra of \mathfrak{g}' . By (3-iii) (see also (1.33)), we have $(h_{\mu^\vee}, h_{\mu^\vee}) = \frac{4}{(\mu, \mu)} \neq 0$. Hence, by (7.1)-(7.2), we see that $(\cdot, \cdot)|_{\mathfrak{g}'\langle\mu\rangle \times \mathfrak{g}'\langle\mu\rangle}$ is non-degenerate. Using Lemma 7.1, we have $\mathfrak{g}'_{\pm\mu} \subset \mathfrak{g}'(p(\mu))$. By this and (7.3)-(7.4), the statement of this step holds.

Step 2: h_σ ($\sigma \in \mathcal{E}$) and E'_μ ($\mu \in \mathcal{B}$) satisfy (SR1,2,3). This is clear from (3-i) and $E'_\mu \in \mathfrak{g}'_\mu$ (see Step 1).

Step 3: E'_μ ($\mu \in \mathcal{B}$) satisfy (SR8,9). This follows from (3-ii) and (4.19).

Step 4: E'_μ ($\mu \in \mathcal{B}$) satisfy (SR5). Let $\mu, \nu \in \mathcal{B}$ be such that $\mu \neq \nu$ and $\mu + \nu \neq 0$. We show

$$(7.5) \quad x_{\mu, \nu} \mu + \nu \notin R$$

(see (1.29) for $x_{\mu, \nu}$); by (1.3), we may assume $\mu \in \mathcal{B}_+$. Recall from (1.14) that

$$(7.6) \quad R \subset (\mathbb{Z}_+ \Pi \oplus \mathbb{Z}a) \cup (\mathbb{Z}_- \Pi \oplus \mathbb{Z}a).$$

We also recall $\gamma^* = c(\gamma)\gamma + k(\gamma)a$ ($\gamma \in \Pi$) from (1.21).

Case 1: $(\mu, \nu) = 0$, so $x_{\mu, \nu} = 1$. In this case, we have $\mu \in \{\alpha, \alpha^*\}$ and $\nu \in \{\beta, -\beta, \beta^*, -\beta^*\}$ for some $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ and $(\alpha, \beta) = 0$. By (7.6), we may assume $\nu \in \{\beta, \beta^*\}$. By (7.6), $s_\nu(\mu + \nu) = \mu - \nu \notin R$. Then we have (7.5) (cf. (AX4)).

Case 2: $(\mu, \nu) > 0$, so $x_{\mu, \nu} = 1$. Then the following two cases exist:

Case 2-1: $\mu \in \{\alpha, \alpha^*\}$ and $\nu \in \{-\beta, -\beta^*\}$ for some $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$ and $(\alpha, \beta) < 0$. In this case, by (7.6), we have (7.5).

Case 2-2: $\mu, \nu \in \{\alpha, \alpha^*\}$ (with $\mu \neq \nu$) for some $\alpha \in \Pi$. Assume $\mu + \nu \in R$. Notice $\mu + \nu = \alpha + \alpha^* = (c(\alpha) + 1)\alpha + k(\alpha)a$. Then $c(\alpha) = 1$ (by (3.2)) and $g(\alpha) \in \{\emptyset, 2\mathbb{Z}, 4\mathbb{Z}\}$ (by (1.20)). However, since $2\alpha + k(\alpha)a \in R$, by (3.2), we have $g(\alpha) \in \{\mathbb{Z}, 2\mathbb{Z} + 1\}$, contradiction. Then we have (7.5). (This can also follow from Lemma 4.3 (1).)

Case 3: $(\mu, \nu) < 0$, so $x_{\mu, \nu} = 1 - (\mu^\vee, \nu)$. Since $(\mu, -\nu) > 0$, by Case 2, we have $s_\mu(-(x_{\mu, \nu}\mu + \nu)) = \mu - \nu \notin R$. By (AX4) and (1.3), we have (7.5).

Then we have (7.5) for all cases. Hence, by (3-ii), the statement of this step holds.

Step 5: There exist $z_\mu \in \mathbb{C} \setminus \{0\}$ ($\mu \in \mathcal{B}_+$) such that $z_\mu^{\pm 1} E'_{\pm\mu}$ satisfy (SR6,7). Let $\mu \in \mathcal{B}_+$. By (AX3), we can see that $|(R \cup V^0) \cap (\nu + \mathbb{Z}\mu)| < \infty$ for $\nu \in R \cup V^0$ (see also the second equality of (1.14) and notice $\mu \in V^\times$). By (3-ii) (see also (1.30)), we see that $E'_{\pm\mu}$ are locally ad-nilpotent (see Subsection 4.2 for the terminology). Hence we can define $n'_\mu \in \text{Aut}(\mathfrak{g}')$ in the same way as in (4.9) with $E'_{\pm\mu}$ in place of $E_{\pm\mu}$.

Let $(\alpha, \beta, y) \in \mathcal{A}$ (see (1.24)). Let $\alpha', \beta' \in \{\alpha, \beta\}$ be of (4.15). Let $\lambda = s_{(\alpha')^*}(\beta')$. Then $\pm\lambda \in R$ (cf. (1.21), (AX4) and (1.3)). By (3-ii), $\dim \mathfrak{g}'_{\pm\lambda} = 1$. By (4.17) and the same formulas as (4.10), we have $n'_{(\alpha')^*}(E'_{\pm\beta'})$, $n'_{\alpha'}(E'_{\pm(\beta')^*}) \in \mathfrak{g}'_{\pm\lambda}$ and $[n'_{(\alpha')^*}(E'_{\beta'}), n'_{(\alpha')^*}(E'_{-\beta'})] = [n'_{\alpha'}(E'_{(\beta')^*}), n'_{\alpha'}(E'_{-(\beta')^*})]$ ($= h_{\lambda^\vee} \neq 0$) (cf. Step 1). By these facts, we have

$$(7.7) \quad n'_{(\alpha')^*}(E'_{\pm\beta'}) = (c'_{\alpha', \beta'})^{\pm 1} n'_{\alpha'}(E'_{\pm(\beta')^*})$$

for some $c'_{\alpha', \beta'} \in \mathbb{C} \setminus \{0\}$. (Compare (7.7) with (4.18).) Let $c_{\alpha, \beta}$ be $(c'_{\alpha', \beta'})^{-1}$ (resp. $c'_{\alpha', \beta'}$) if $\alpha' = \beta$ (resp. $\alpha' = \alpha$). By (7.7) and (4.16), using the same formulas as in (4.13), we have:

$$(7.8) \quad (\pm 1)^{c(\alpha)+1} c(\alpha) (\text{ad} E_{\pm\alpha})^y E_{\pm\beta} = c_{\alpha, \beta}^{\pm 1} (\text{ad} E_{\pm\alpha})^{c(\alpha)y} E_{\pm\beta^*}.$$

Recall from (3.29) (or Remark 1.2 (3)) that $A(\Pi)$ (see (1.15) and (1.45)) is not $A_l^{(1)}$, so the Dynkin diagram associated with $A(\Pi)$ (or $(A(\Pi), \emptyset)$) is not a loop (cf. Section 9). Then by (7.8), we can see that the statement of this step holds.

Step 6: An epimorphism $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ with $f(h_\sigma) = h_\sigma$ ($\sigma \in \mathcal{E}$) exists. By Steps 1-5, we can define a homomorphism $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ by $f(h_\sigma) = h_\sigma$ ($\sigma \in \mathcal{E}$) and $f(E_{\pm\mu}) = z_\mu^{\pm 1} E'_{\pm\mu}$ ($\mu \in \mathcal{B}_+$). By (3-ii), (3-iv), Lemma 4.4 (3) and the same formulas as in (4.8) and (4.10), we see that \mathfrak{g}' is generated by h_σ ($\sigma \in \mathcal{E}$) and $z_\mu^{\pm 1} E'_{\pm\mu}$ ($\mu \in \mathcal{B}_+$). Hence f is an epimorphism, as desired. This completes the proof. \square

8 Proofs of well-known facts

Here we give proofs of Theorem 2.1 and Lemmas 2.1, 2.4, 2.5, 4.2, 4.7 and 7.1. We also give Remark 8.1, which is detail of (5.2).

Proof of Lemma 2.1. (1) This follows from [6, Theorem 10.3 (d)].

(2) This follows from [6, Lemma 10.4C] and (1).

(3) (*Strategy.* We use a (closed) fundamental Weyl chamber $\overline{\mathfrak{C}(\Pi)} = \{v \in V \mid (v, \alpha_i) \geq 0 \text{ for all } 1 \leq i \leq l\}$.) Recall $W_\Pi = W$ from (1). Define $f \in V^*$ by $f(\alpha) = 1$ ($\alpha \in \Pi$). Using induction on $f(\mu) \in \mathbb{Z}$ with $\mu \in R$ and noticing $s_\alpha(v) = v - (\alpha^\vee, v)\alpha$ ($\alpha \in \Pi$), we can see that for any $\gamma \in R$, there exists $w \in W_\Pi$ such that $w(\gamma) \in \overline{\mathfrak{C}(\Pi)}$ and $w(\gamma) - \gamma \in \mathbb{Z}_+\Pi$. Then, by [6, Lemmas 10.3B] and (1)-(2), we can see that (3) holds (especially $w(\gamma)$ is θ_{sh} , θ_{lg} or θ_{lg}). \square

Let R be an irreducible finite root system with a base Π . By checking directly (and using [6, §12 Table 2]), we have

$$(8.1) \quad (\mu, \nu) > 0 \text{ for } \mu, \nu \in \Theta(R, \Pi).$$

(see (2.1) for $\Theta(R, \Pi)$).

Proof of Theorem 2.1. (1) (*Strategy.* We use a linear map $f : V \rightarrow \mathbb{R}$ (i.e., $f \in V^*$) such that $f(\alpha_i) = 1$ ($1 \leq i \leq l$) and $f(\delta')$ is sufficiently large (see (8.5)). Let Π^f be the subset of R formed by the elements $\beta \in R$ satisfying the condition that $f(\beta) > 0$ and β is not expressed as the summation of more than one elements β' of R with $f(\beta') > 0$ (see (8.7)). We show that

Π^f is a base of R satisfying (1-1)-(1-4). It is easy to see that $\Pi' \subset \Pi^f$ and $R = (R \cap \mathbb{Z}_+ \Pi^f) \cup (R \cap \mathbb{Z}_- \Pi^f)$. We show $|\Pi^f| = l + 1$ by using (8.1).

We proceed in steps.

Step 1 (Definition of f). Notice that for $\mathbb{X} = \mathbb{R}$ or \mathbb{Z} ,

$$(8.2) \quad \mathbb{X}R = \mathbb{X}\delta' \oplus (\oplus_{i=1}^l \mathbb{X}\alpha_i)$$

(see (1.4)). Assume that $(\alpha_i, \alpha_i) \leq (\alpha_{i+1}, \alpha_{i+1})$ for $1 \leq i \leq l - 1$. If $l \geq 2$, we assume α_1 to be such that there exists a unique $j \in \{2, \dots, l\}$ such that $(\alpha_1, \alpha_j) \neq 0$. Let

$$(8.3) \quad R' = \begin{cases} W_{\Pi'}(\Pi' \cup \{2\alpha_1\}) & \text{if } l = 1, \\ W_{\Pi'}(\Pi' \cup \{2\alpha_1\}) & \text{if } l \geq 2 \text{ and } 2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2), \\ W_{\Pi'}\Pi' & \text{otherwise.} \end{cases}$$

Using [6, Theorem 10.3 (c) (and §12 Exercise 3)], we can see that $W_{\Pi'}\Pi'$ and R' are irreducible finite root systems with the base Π' . We also see that $\pi(W_{\Pi'}\Pi') \subset \pi(R) \subset \pi(R')$. Then by (8.2), we have

$$(8.4) \quad R \subset R' + \mathbb{Z}\delta'.$$

Define $f \in V^*$ by

$$(8.5) \quad f(\alpha_i) = 1 \quad (1 \leq i \leq l) \quad \text{and} \quad f(\delta') = 3M,$$

where $M = \max\{|f(\gamma)| \mid \gamma \in R'\}$ (notice $|R'| < \infty$). It follows from (8.4) that $f(\beta) \neq 0$ for $\beta \in R$.

Step 2 (Definition of Π^f). Let $R^{f,+} = \{\beta \in R \mid f(\beta) > 0\}$. By (8.5), we have

$$(8.6) \quad R^{f,+} = R \cap ((R' \cap \mathbb{Z}_+ \Pi') \cup (\cup_{m=1}^{\infty} (m\delta' + R'))).$$

Let Π^f be a subset of R formed by the elements $\beta \in R^{f,+}$ satisfying the condition that there exist no $\beta_1, \dots, \beta_r \in R^{f,+}$ with $r \geq 2$ such that $\beta = \beta_1 + \dots + \beta_r$; namely,

$$(8.7) \quad \Pi^f = R^{f,+} \setminus \left(\bigcup_{r=2}^{\infty} \left\{ \sum_{i=1}^r \beta_i \mid \beta_i \in R^{f,+} \right\} \right).$$

By (8.6), we have

$$(8.8) \quad \Pi' \subset \Pi^f.$$

Notice $\mathbb{Z}\Pi' \neq \mathbb{Z}R$ (by (8.2)). Then we have

$$(8.9) \quad \mathbb{Z}\Pi^f = \mathbb{Z}R, \quad R = (R \cap \mathbb{Z}_+\Pi^f) \cup (R \cap \mathbb{Z}_-\Pi^f) \text{ and } |\Pi^f| \geq |\Pi'| + 1.$$

(As mentioned in Strategy, we show that Π^f is a base of R .)

Step 3 (If $\beta \in \Pi^f/\Pi'$, then we have $\pi(\beta) \in \Theta(\pi(R), \pi(\Pi'))$ (see (2.1) for $\Theta(\pi(R), \pi(\Pi'))$)). Let $\beta \in \Pi^f/\Pi'$ (see also (8.8)-(8.9)). We show that β is expressed as

$$(8.10) \quad \beta = m\delta' - \theta$$

for some $m \in \mathbb{N}$ and some θ with

$$(8.11) \quad \theta \in \Theta(R', \Pi')$$

(see (2.1) for $\Theta(R', \Pi')$). By (8.6), since $\Pi^f \subset R^{f,+}$, we have

$$(8.12) \quad \beta = m\delta' + \mu$$

for some $m \in \mathbb{N}$ and $\mu \in R'$. Let $\theta \in \Theta(R', \Pi') \cap W_{\Pi'}.\mu$, where we recall from Lemma 2.1 (2)-(3) that $|\Theta(R', \Pi') \cap W_{\Pi'}.\mu| = 1$. Notice $\{\mu, -\mu, \theta, -\theta\} \subset W_{\Pi'}.\mu$ (cf. Lemma 2.1 (2)). Then $m\delta' - \theta \in R$ since $m\delta' - \theta \in m\delta' + W_{\Pi'}.\mu = W_{\Pi'}.(m\delta' + \mu) = W_{\Pi'}.\beta \subset R$. By Lemma 2.1 (3), we have $\theta + \mu = \theta - (-\mu) \in \mathbb{Z}_+\Pi'$. Since $m\delta' - \theta \in R^{f,+}$ (cf. (8.6)), $\beta = (m\delta' - \theta) + (\theta + \mu)$ and $\beta \in \Pi^f$, we have $\theta + \mu = 0$ and (8.10), as desired.

Step 4 ($|\Pi^f| = l + 1$). We show

$$(8.13) \quad |\Pi^f \setminus \Pi'| = 1, \text{ i.e., } |\Pi^f| = l + 1$$

(see also (8.8)-(8.9)).

Assume $|\Pi^f \setminus \Pi'| > 1$. Let $\beta_1, \beta_2 \in \Pi^f \setminus \Pi'$ and assume $\beta_1 \neq \beta_2$. Assume $(\beta_1, \beta_1) \leq (\beta_2, \beta_2)$. Then, by (8.1) and (8.10)-(8.11), we see that

$$(\beta_2^\vee, \beta_1) = \begin{cases} 1 & \text{if } \pi(\beta_1) \neq \pi(\beta_2), \\ 2 & \text{if } \pi(\beta_1) = \pi(\beta_2). \end{cases}$$

Assume $(\beta_2^\vee, \beta_1) = 1$. Then, since $\pm(\beta_1 - \beta_2) = s_{\beta_2}(\pm\beta_1) \in R$ (by (AX4) and (1.3)), we have $\beta_1 - \beta_2 \in R^{f,+}$ or $\beta_2 - \beta_1 \in R^{f,+}$. This contradicts the fact $\beta_1, \beta_2 \in \Pi^f$ since $\beta_1 = \beta_2 + (\beta_1 - \beta_2)$ and $\beta_2 = \beta_1 + (\beta_2 - \beta_1)$.

Assume $(\beta_2^\vee, \beta_1) = 2$, so $\pi(\beta_1) = \pi(\beta_2)$. By (8.10), there exist $n_1, n_2 \in \mathbb{N}$ and $\theta \in \Theta(R', \Pi')$ (see (8.11)) such that

$$\beta_i = n_i \delta - \theta \quad (i \in \{1, 2\})$$

(so $\beta_2 - \beta_1 = (n_2 - n_1)\delta$). Assume $n_1 < n_2$. Notice that for $i \in \{1, 2\}$ and $r \in \mathbb{Z}$,

$$(8.14) \quad \begin{aligned} R &\ni (s_{\beta_2} s_{\beta_1})^r(\beta_i) \quad (\text{by (AX4)}) \\ &= (n_i + 2r(n_2 - n_1))\delta - \theta \\ &= \begin{cases} (n_2 + (2r - 1)(n_2 - n_1))\delta - \theta & \text{if } i = 1, \\ (n_2 + 2r(n_2 - n_1))\delta - \theta & \text{if } i = 2. \end{cases} \end{aligned}$$

Hence

$$(8.15) \quad (n_2 + r(n_2 - n_1))\delta - \theta \in R \quad \text{for all } r \in \mathbb{Z}.$$

Let $n_3 \in \mathbb{Z}_+$ and $t \in \mathbb{N}$ be such that $0 \leq n_3 < n_2 - n_1$ and $n_2 = t(n_2 - n_1) + n_3$. Assume $n_3 = 0$. By (8.15), $\{-\theta, (n_2 - n_1)\delta - \theta\} \subset R$. Hence, by (8.6) (and (1.3)), $\{\theta, (n_2 - n_1)\delta - \theta\} \subset R^{f,+}$. Notice $t \geq 2$ (since $0 < n_1 < n_2$ and $n_3 = 0$). Since $\beta_2 = t((n_2 - n_1)\delta - \theta) + (t - 1)\theta$, we have $\beta_2 \notin \Pi^f$, contradiction. Assume $n_3 > 0$. Notice $2n_3 < n_2$ (since $2n_3 < (n_2 - n_1) + n_3 \leq t(n_2 - n_1) + n_3 = n_2$). Let $\beta_3 = n_3\delta - \theta$. By (8.15), $\beta_3 \in R$. By (8.6), $\beta_3 \in R^{f,+}$. Notice $\beta_2 - 2\beta_3 = s_{\beta_3}(\beta_2) \in R$ (by (AX4)). Then by (8.6), we have

$$\beta_2 - 2\beta_3 = (n_2 - 2n_3)\delta + \theta \in R^{f,+}.$$

Since $\beta_2 = (\beta_2 - 2\beta_3) + 2\beta_3$, we have $\beta_2 \notin \Pi^f$, contradiction. Hence $|\Pi^f| = l + 1$, as desired.

Step 5 (Π^f is a base satisfying (1-1)-(1-4)). Let α_0 be $\beta = m\delta' - \theta$ of (8.10). Then $\Pi^f = \Pi' \cup \{\alpha_0\}$, where we notice (8.8) and (8.13). It is clear that the elements of Π^f are linearly independent (cf. (8.2)). Hence, by (8.9), Π^f is a base of R (cf. (1.6)). Since $\mathbb{Z}\Pi' \oplus \mathbb{Z}\delta' = \mathbb{Z}\Pi' \oplus \mathbb{Z}\alpha_0$ (by (8.2) and (8.9)), we have $m = 1$. Hence (1-1) and (1-2) hold. By (8.11) and Lemma 2.2 (1), we have (1-3). Since $((2\alpha_i)^\vee, \alpha_j) = \frac{1}{2}(\alpha_i^\vee, \alpha_j)$, we can easily see that (1-4) holds. This completes the proof of (1).

(2) This can be proved by case-by-case checking. (See also [11, Appendixes 1 and 2].)

(3) (*Strategy*. We use an induction on $|(R \cap \mathbb{Z}_+\Pi_1 \cap \mathbb{Z}_-\Pi_2) \setminus 2R|$ (this may be assumed to be finite). We also use the fact that for all $\alpha \in \Pi_1$, $s_\alpha((R \cap \mathbb{Z}_+\Pi_1) \setminus 2R) = \{-\alpha\} \cup (((R \cap \mathbb{Z}_+\Pi_1) \setminus 2R) \setminus \{\alpha\})$.)

We may assume Π_1 to be the base $\Pi' \cup \{\alpha_0 = \alpha_0(R, \Pi', \delta') = \delta' - \theta\}$ of R given by (1). Define $h \in V^*$ by $h(\beta) = 1$ ($\beta \in \Pi_2$). Then $h(R) \subset \mathbb{Z} \setminus \{0\}$. By the same formula as in (8.14), we have $|\{(s_\theta s_{\alpha_0})^r(\alpha_0) \in R | r \in \mathbb{Z}\}| = \infty$ (notice that $(s_\theta s_{\alpha_0})^r(\alpha_0) \in R$ (by (AX4)) since $s_\theta = s_{\frac{1}{2}\theta}$ and $\theta \in R \cup 2R$ (see (8.11) and (8.3))). Hence $|R| = \infty$, which implies $|\check{h}(R)| = \infty$. Hence, by (8.4), since $|R'| < \infty$ (R' is an irreducible finite root system), we have $h(\delta') \neq 0$. We may assume

$$(8.16) \quad h(\delta') > 0$$

(otherwise, we replace Π_2 with $-\Pi_2$). Let

$$m(\Pi_1, \Pi_2) = |(R \cap \mathbb{Z}_+ \Pi_1 \cap \mathbb{Z}_- \Pi_2) \setminus 2R|.$$

Since $\alpha_0 = \delta' - \theta$, we have $R \cap \mathbb{Z}_+ \Pi_1 \subset R' + \mathbb{Z}_+ \delta'$ (cf. (8.4)). Hence, since $|R'| < \infty$, by (8.16), we have

$$m(\Pi_1, \Pi_2) = |\{\beta \in (R \cap \mathbb{Z}_+ \Pi_1) \setminus 2R | h(\beta) < 0\}| < \infty.$$

We use induction on $m(\Pi_1, \Pi_2)$; if $m(\Pi_1, \Pi_2) = 0$, then, by (1.6), $R \cap \mathbb{Z}_+ \Pi_1 = R \cap \mathbb{Z}_+ \Pi_2$, so $\Pi_1 = \Pi_2$. Assume $m(\Pi_1, \Pi_2) > 0$. Then there exists $\alpha \in \Pi_1$ such that $\alpha \in \mathbb{Z}_- \Pi_2$ (notice that $R \subset \mathbb{Z}_- \Pi_2 \cup \mathbb{Z}_+ \Pi_2$). By (1.6) (and (1.3)), we see

$$(8.17) \quad s_\alpha((R \cap \mathbb{Z}_+ \Pi_1) \setminus 2R) = \{-\alpha\} \cup (((R \cap \mathbb{Z}_+ \Pi_1) \setminus 2R) \setminus \{\alpha\}).$$

Then we have

$$\begin{aligned} & m(\Pi_1, s_\alpha(\Pi_2)) \\ &= |(R \cap \mathbb{Z}_+ \Pi_1 \cap \mathbb{Z}_- s_\alpha(\Pi_2)) \setminus 2R| \\ &= |s_\alpha((R \cap \mathbb{Z}_+ \Pi_1 \cap \mathbb{Z}_- s_\alpha(\Pi_2)) \setminus 2R)| \\ &= |(s_\alpha(R \cap \mathbb{Z}_+ \Pi_1) \cap \mathbb{Z}_- \Pi_2) \setminus 2R| \\ &= m(\Pi_1, \Pi_2) - 1 \quad (\text{by (8.17) since } s_\alpha(\alpha) = -\alpha \notin \mathbb{Z}_- \Pi_2). \end{aligned}$$

Then, by the induction, we see that there exists $w \in W_\Pi$ such that $w(\Pi_2) = \Pi_1$, as desired.

(4) This follows from (1) and (3). \square

Proof of Lemma 2.4. (*Strategy.* We show (4) by induction on the height with respect to Π . (1) and (2) follow from (4). (3) follows from (1).)

(4) Define $h \in V^*$ by $h(\alpha) = 1$ ($\alpha \in \Pi$). Let $\mu \in \mathbb{Z}_+S \setminus \mathbb{Z}\delta$ (we may assume this). We use induction on $h(\mu)$; if $h(\mu) = 1$, then $\mu \in S$. Assume $h(\mu) \geq 2$. We may assume $\mu \notin \cup_{\alpha \in S} \mathbb{N}\alpha$. Since $(\mu, \mu) > 0$ (cf. (2.2)), there exists $\alpha \in S$ such that $(\alpha, \mu) > 0$. Then we see that if $s_\alpha(\mu) (= \mu - (\alpha^\vee, \mu)\alpha) \notin \mathbb{Z}S \setminus (\mathbb{Z}_+S \cup \mathbb{Z}_-S)$, then $s_\alpha(\mu) \in \mathbb{Z}_+S \setminus \mathbb{Z}\delta$ and $0 < h(s_\alpha(\mu)) < h(\mu)$. Thus, by the induction, we can see that (4) hold.

(1) and (2) These follows from (2.13), (2.10) and the fact $R \subset \mathbb{Z}_+\Pi \cup \mathbb{Z}_-\Pi$ (see (1.6)).

(3) For β and w of (2.11), we have $s_\beta = w^{-1}s_{w(\beta)}w \in W_S$. Then (3) follows from this fact and the definition of $W_{R \cap \mathbb{R}S}$. \square

Proof of Lemma 2.5. (*Strategy.* We use Theorem 2.1 (4) (4-4).)

(II) \Rightarrow (I) This follows from the fact that

$$s_{\gamma_i}s_{\gamma_{i+1}}(\gamma_i) = s_{\gamma_i}(\gamma_i + \gamma_{i+1}) = \gamma_{i+1} \quad (1 \leq i \leq l-1).$$

(I) \Rightarrow (II) Notice that $(\alpha, \alpha) = (\beta, \beta)$ since $w(\alpha) = \beta$. Assume that γ_i 's of (II) do not exist. Then, by Theorem 2.1 (4) (4-4), seeing Section 9, we see that $A(\Pi)$ is $A_1^{(1)}$ ($l = 1$), $C_l^{(1)}$ or $D_{l+1}^{(2)}$ (see also (1.45)) and α and β correspond to the two end dots of the Dynkin diagram defined for $A(\Pi)$ (or $(A(\Pi), \emptyset)$). Hence, letting $\Pi' = \Pi \setminus \{\beta\}$, we can see that $W.\alpha = W_{\Pi'}.\alpha + 2\mathbb{Z}\delta$ and $W.\beta = W_{\Pi'}.\alpha + (2\mathbb{Z} + 1)\delta$, where $\delta = \delta(\Pi)$ (see (2.2)). This contradicts the fact that $w(\alpha) = \beta$. Hence (II) holds. This completes the proof. \square

Proof of Lemma 4.2. (*Strategy.* Clearly (4) holds. Since $\bar{E}_{\pm\mu}$ are locally ad-nilpotent, by (4.20) and the same formula as the first one of (4.6) and by Theorem 2.1 (2), we see that there exists an affine root system R'_P with the base P such that $R'_P \cap 2P = 2P'$. By Lemma 2.4 (1), we have $R'_P = R_P$, so (1)-(3) hold.

By (4.20), we have the triangular decomposition $\mathfrak{g}_P = \mathfrak{n}_P^+ \oplus \mathfrak{h} \oplus \mathfrak{n}_P^-$, where \mathfrak{n}_P^\pm are the sub-Lie superalgebras generated by $\bar{E}_{\pm\mu}$ ($\mu \in P$) (see (8.18)). Using the triangular decomposition, we can see that for each $\mu \in P$, $\dim(\mathfrak{g}_P)_\mu = 1$ and $\dim(\mathfrak{g}_P)_{2\mu} = \delta_{1, \bar{p}(\mu)}$. We have $\dim(\mathfrak{g}_P)_{w(\sigma)} = \dim(\mathfrak{g}_P)_\sigma$ for $w \in W_P$ (see (8.24)). Then (5)-(8) follows from these facts and Lemma 2.4 (1)-(2).)

We proceed in steps.

Step 1 ((1)-(4) holds). Clearly (4) holds. Since $\bar{E}_{\pm\mu}$ are locally ad-nilpotent, by (4.20) and the same formula as the first one of (4.6), we have $(\mu^\vee, \nu) \in 2\mathbb{Z}_-$ for $\mu \in P'$ and $\nu \in P$. By Theorem 2.1 (2), there exists

an affine root system R'_P with the base P such that $R'_P \cap 2P = 2P'$. By Lemma 2.4 (1), we have $R'_P = W_P \cdot (P \cup 2P')$, so $R'_P = R_P$ by (4.21). Hence (1)-(3) hold.

Step 2 ((8) and $\mathfrak{g}_P = \mathfrak{n}_P^+ \oplus \mathfrak{h} \oplus \mathfrak{n}_P^-$ hold). Let \mathfrak{n}_P^+ (resp. \mathfrak{n}_P^-) be the sub-Lie superalgebra generated by \bar{E}_μ (resp. $\bar{E}_{-\mu}$) with all $\mu \in P$. By (4.20) and Lemma 4.1, we have

$$(8.18) \quad \mathfrak{g}_P = \mathfrak{n}_P^+ \oplus \mathfrak{h} \oplus \mathfrak{n}_P^-.$$

Hence, since $\delta(P) \in \text{NII}(\subset \mathbb{Z}_+\Pi)$, we see that

$$(8.19) \quad \mathfrak{g}_P = \mathfrak{h} \bigoplus_{\lambda \in (\mathbb{Z}_+P \cup \mathbb{Z}_-P) \setminus \mathbb{Z}\delta(P)} (\mathfrak{g}_P)_\lambda \bigoplus_{m \in \mathbb{Z} \setminus \{0\}} (\mathfrak{g}_P)_{m\delta(P)}$$

and that (8) holds. In the same way as that for (4.8), using Lemma 4.1, we have

$$(8.20) \quad \bar{E}_{\pm\mu} \neq 0 \ (\mu \in P), \quad [\bar{E}_{\pm\nu}, \bar{E}_{\pm\nu}] \neq 0 \ (\nu \in P').$$

By (8.18) and (8.20), we have

$$(8.21) \quad \begin{cases} (\mathfrak{g}_P)_\mu = \mathbb{C}\bar{E}_\mu \text{ and } \dim(\mathfrak{g}_P)_\mu = 1 & \text{if } \mu \in P, \\ \dim(\mathfrak{g}_P)_{2\mu} = 0 & \text{if } \mu \in P \setminus P', \\ (\mathfrak{g}_P)_{2\mu} = \mathbb{C}[\bar{E}_\mu, \bar{E}_\mu] \text{ and } \dim(\mathfrak{g}_P)_{2\mu} = 1 & \text{if } \mu \in P', \\ \dim(\mathfrak{g}_P)_{x\mu} = 0 & \text{if } \mu \in P, x \in \mathbb{R}_{>0} \setminus \{1, 2\}. \end{cases}$$

where $\mathbb{R}_{>0} = \{y \in \mathbb{R} | y > 0\}$.

Step 3 ((5) and (6) hold). Let $\mu \in P$. Since $\bar{E}_{\pm\mu}$ are locally ad-nilpotent, we can define $\bar{n}_\mu \in \text{Aut}(\mathfrak{g})$ in the same way as in (4.9) with $\bar{E}_{\pm\mu}$ (in place of $E_{\pm\mu}$). It is clear that $\bar{n}_\mu(\mathfrak{g}_P) = \mathfrak{g}_P$. As in (4.10), we see that for $\sigma \in \mathcal{E}$,

$$(8.22) \quad \bar{n}_\mu((\mathfrak{g}_P)_\sigma) = (\mathfrak{g}_P)_{s_\mu(\sigma)},$$

which implies

$$(8.23) \quad \dim(\mathfrak{g}_P)_{s_\mu(\sigma)} = \dim(\mathfrak{g}_P)_\sigma.$$

By (8.23), for $w \in W_P$ and $\sigma \in \mathcal{E}$, we have

$$(8.24) \quad \dim(\mathfrak{g}_P)_{w(\sigma)} = \dim(\mathfrak{g}_P)_\sigma.$$

Then (5) and (6) follows from (1)-(3), Lemma 2.4 (1)-(2), (8.19), (8.21) and (8.24).

Step 4 ((7) holds). Let $\lambda \in R_P \cap \frac{1}{2}R_P$. By (4.21), $w(\lambda) \in P \cup 2P'$ for some $w \in W_P$. Since $2w(\lambda) \in R_P$, we have $w(\lambda) \notin P \setminus P'$ by (3) and we have $w(\lambda) \notin 2P'$ by (1.3). Hence $w(\lambda) \in P'$, so $R_P \cap \frac{1}{2}R_P \subset W_P.P'$. On the other hand, $W_P.P' \subset R_P \cap \frac{1}{2}R_P$ is clear from (4.21). Hence, by (4.21), we have $R_P \setminus \frac{1}{2}R_P \subset W_P.(P \setminus P') \cup 2P'$. Notice that $\bar{n}_\mu(\mathfrak{g}(i)) = \mathfrak{g}(i)$ ($\mu \in P$). Then (7) follows from (8.21) and (8.22). This completes the proof. \square

Proof of Lemma 4.7. (Strategy. We use the Jacobi identity and induction on r .)

We use induction on r . The case of $r = 1$ is trivial. The case of $r = 2$ is also trivial since $[\mathfrak{a}_1, \mathfrak{a}_2] = [\mathfrak{a}_2, \mathfrak{a}_1]$.

We assume $r \geq 3$. Then we have the following:

$$\begin{aligned}
& (\text{ada}_{\mathfrak{a}_1}) \cdots (\text{ada}_{\mathfrak{a}_{r-1}})(\mathfrak{a}_r) \\
& \subset (\text{ada}_{\mathfrak{a}_2})(\text{ada}_{\mathfrak{a}_1})(\text{ada}_{\mathfrak{a}_3}) \cdots (\text{ada}_{\mathfrak{a}_{r-1}})(\mathfrak{a}_r) + (\text{ad}[\mathfrak{a}_1, \mathfrak{a}_2])(\text{ada}_{\mathfrak{a}_3}) \cdots (\text{ada}_{\mathfrak{a}_{r-1}})(\mathfrak{a}_r) \\
& \quad (\text{by the Jacobi identity (cf. Subsection 1.5)}) \\
& \subset \left(\sum_{\tau \in S_{r-2}} (\text{ada}_{\mathfrak{a}_2})(\text{ada}_{\tau(1)+2}) \cdots (\text{ada}_{\tau(r-2)+2})(\mathfrak{a}_1) \right) \\
& \quad + \left(\sum_{\tau \in S_{r-2}} (\text{ada}_{\tau(1)+2}) \cdots (\text{ada}_{\tau(r-2)+2})([\mathfrak{a}_2, \mathfrak{a}_1]) \right) \\
& \quad (\text{by induction and } [\mathfrak{a}_1, \mathfrak{a}_2] = [\mathfrak{a}_2, \mathfrak{a}_1]) \\
& \subset \sum_{\sigma \in S_{r-1}} (\text{ada}_{\sigma(1)+1}) \cdots (\text{ada}_{\sigma(r-1)+1})(\mathfrak{a}_1).
\end{aligned}$$

This completes the proof. \square

Remark 8.1. (On (SR4) in the proofs of Lemmas 5.5-5.9 (Detail of (5.2)))

Let $\psi_i \rightarrow \mathbb{L}$ be the homomorphisms of Lemmas 5.5-5.9. In the proof of Lemmas 5.5-5.9, to check the equations

$$(8.25) \quad [\psi_i(E_{\alpha^*}), \psi_i(E_{-\alpha^*})] = \psi_i(h_{(\alpha^*)^\vee}) \quad (\alpha \in \Pi),$$

we can use the following argument, where $\alpha^* = c(\alpha)\alpha + k(\alpha)a$ (see (3.24) and (1.21)). Assume that we have checked that $\psi_i(h_\sigma)$ ($\sigma \in \mathcal{E}$) and $\psi_i(E_\mu)$ ($\mu \in \mathcal{B}$) satisfy (SR1-3) and (SR4) except for (8.25). We also assume that

for all $\alpha \in \Pi$, we have checked that

$$(8.26) \quad [\psi_i(E_{\alpha^*}), \psi_i(E_{-\alpha^*})] = \frac{1}{c(\alpha)} \psi_i(h_{\alpha^\vee}) + x_\alpha \psi_i(h_\alpha)$$

for some $x_\alpha \in \mathbb{C}$. Let $y_\alpha = \frac{x_\alpha c(\alpha)^2(\alpha, \alpha)}{2k(\alpha)}$. Since $h_{(\alpha^*)^\vee} = \frac{1}{c(\alpha)} h_{\alpha^\vee} + \frac{2k(\alpha)}{c(\alpha)^2(\alpha, \alpha)} \psi_i(h_\alpha)$, we need to show $y_\alpha = 1$. By the same way as that for (6.11), we can have a super-symmetric invariant form $(,)''$ on \mathbb{L} such that $(\psi_i(h_\alpha), \psi_i(h_\alpha))'' = (\psi_i(h_{\Lambda_\alpha}), \psi_i(h_\alpha))'' = 0$ ($\alpha \in \Pi$) and $(\psi_i(h_{\Lambda_\alpha}), \psi_i(h_\alpha))'' = 1$ (we can easily see this; we do not need an argument similar to that in Step 2 of Proof of Lemma 6.1). Let $\alpha, \beta \in \Pi$ be such that $\alpha \neq \beta$ and $(\alpha, \beta) \neq 0$. Then we have

$$\begin{aligned} y_\alpha &= \frac{c(\alpha)^2(\alpha, \alpha)}{2k(\alpha)} (\psi_i(h_{\Lambda_\alpha}), [\psi_i(E_{\alpha^*}), \psi_i(E_{-\alpha^*})])'' \quad (\text{by (8.26)}) \\ &= \frac{c(\alpha)^2(\alpha, \alpha)}{2k(\alpha)} ([\psi_i(h_{\Lambda_\alpha}), \psi_i(E_{\alpha^*})], \psi_i(E_{-\alpha^*}))'' \\ &= \frac{c(\alpha)^2(\alpha, \alpha)}{2} (\psi_i(E_{\alpha^*}), \psi_i(E_{-\alpha^*}))'' \\ &= \frac{c(\alpha)(\alpha, \alpha)}{2(\beta^\vee, \alpha)} ([\psi(h_{\beta^\vee}), \psi_i(E_{\alpha^*})], \psi_i(E_{-\alpha^*}))'' \\ &= \frac{c(\alpha)(\alpha, \alpha)}{2(\beta^\vee, \alpha)} (\psi(h_{\beta^\vee}), [\psi_i(E_{\alpha^*}), \psi_i(E_{-\alpha^*})])'' \\ &= \frac{(\alpha, \alpha)}{2(\beta^\vee, \alpha)} (\psi_i(h_{\beta^\vee}), \psi_i(h_{\alpha^\vee}))'' \quad (\text{by (8.26)}) \\ &= \frac{(\alpha, \alpha)(\beta, \beta)}{4(\beta, \alpha)} (\psi_i(h_{\beta^\vee}), \psi_i(h_{\alpha^\vee}))'' \\ &= y_\beta \quad (\text{by symmetry}). \end{aligned}$$

Thus, once we check (8.25) (or $y_\alpha = 1$) for some α , we see that (8.25) hold for all α .

Proof of Lemma 7.1. (*Strategy.* By (i), in a standard way (see [8, Proof of Theorem 2.2]), we conclude that for $1 \leq m \leq r$, there exist $X_{\pm m} \in \mathfrak{a}_{\pm m}$ such that $[X_m, X_{-m}] = H$. Then we can easily prove the lemma.)

We may assume

$$(8.27) \quad (H, H) = 2.$$

We proceed in steps.

Step 1 (The elements $X_{\pm m}$ in Strategy exist). By (iii), for each $-r \leq m \leq r$, there exists $p_m \in \{0, 1\}$ such that $\mathfrak{a}_m \subset \mathfrak{a}(p_m)$. For $-r \leq m \leq r$, let $X_m \in \mathfrak{a}_m \setminus \{0\}$ (i.e., $\mathbb{C}X_m = \mathfrak{a}_m$ (see (iii))); we assume $X_0 = H$. In the same way as that for (6.8), we can see that if $i + j \neq 0$, $(\mathfrak{a}_i, \mathfrak{a}_j) = \{0\}$. Hence, since $(,)$ is non-degenerate (see (i)), we may assume that for $1 \leq m \leq r$, $(X_m, X_{-m}) = \frac{1}{m}$, which implies $(H, H - [X_m, X_{-m}]) = 0$ (see (8.27)). Hence, since $\dim \mathfrak{a}_0 = 1$ (see (iii)), we have

$$(8.28) \quad [X_m, X_{-m}] = H \quad (1 \leq m \leq r).$$

Step 2 ($p_{\pm r} = 0$). By (8.28), we have

$$(8.29) \quad p_m = p_{-m} \quad (1 \leq m \leq r).$$

By (8.28) and (8.29), we have

$$(8.30) \quad ([X_m, X_m], [X_{-m}, X_{-m}]) = -\delta_{p_m 1} \cdot \frac{4}{m} \quad (1 \leq m \leq r).$$

By (8.30) and (ii)-(iii), we see that for $1 \leq m \leq r$,

$$(8.31) \quad p_m = 1 \implies m = 1, r = 2 \text{ and } \mathfrak{a}_{\pm 2} = \mathbb{C}[X_{\pm 1}, X_{\pm 1}].$$

By (8.31) and (8.29), we have

$$(8.32) \quad p_r = p_{-r} = 0$$

(notice $r \in \{1, 2\}$).

Step 3 (if $r = 2$, then $p_1 = 1$). We show that

$$(8.33) \quad r = 2 \implies p_1 = 1.$$

Assume $r = 2$. By (ii) and (8.28), we have

$$(8.34) \quad [X_1, [X_{-1}, X_2]] = [H, X_2] = 4X_2 \neq 0.$$

By (iii) and (8.34), there exists $x \in \mathbb{C} \setminus \{0\}$ such that $X_1 = x[X_{-1}, X_2]$. Hence, by (8.34), we see $[X_1, X_1] \neq 0$, whence $p_1 = 1$. Then we have (8.33), as desired.

Step 4 (The lemma holds). Set $E = X_1$ and $F = X_{-1}$. Then, by (8.28), (8.29), (8.31), (8.32) and (8.33), we see that the lemma holds. \square

9 Dynkin diagrams of the affine root systems

Here we give the Dynkin diagrams defined for (A, τ) of Theorem 2.1 (2). We describe them in the same manner as in [10, Table 1-4]; notice that if $i \notin \tau$ (resp. $i \in \tau$), then the i -th dot is white (resp. black). The names of them are also the same as in [10, Table 1-4]. As for (1.45), the Dynkin diagram associated with $A(\Pi)$ is the one defined for $(A(\Pi), \emptyset)$.

(i) The case of $l = 1$:

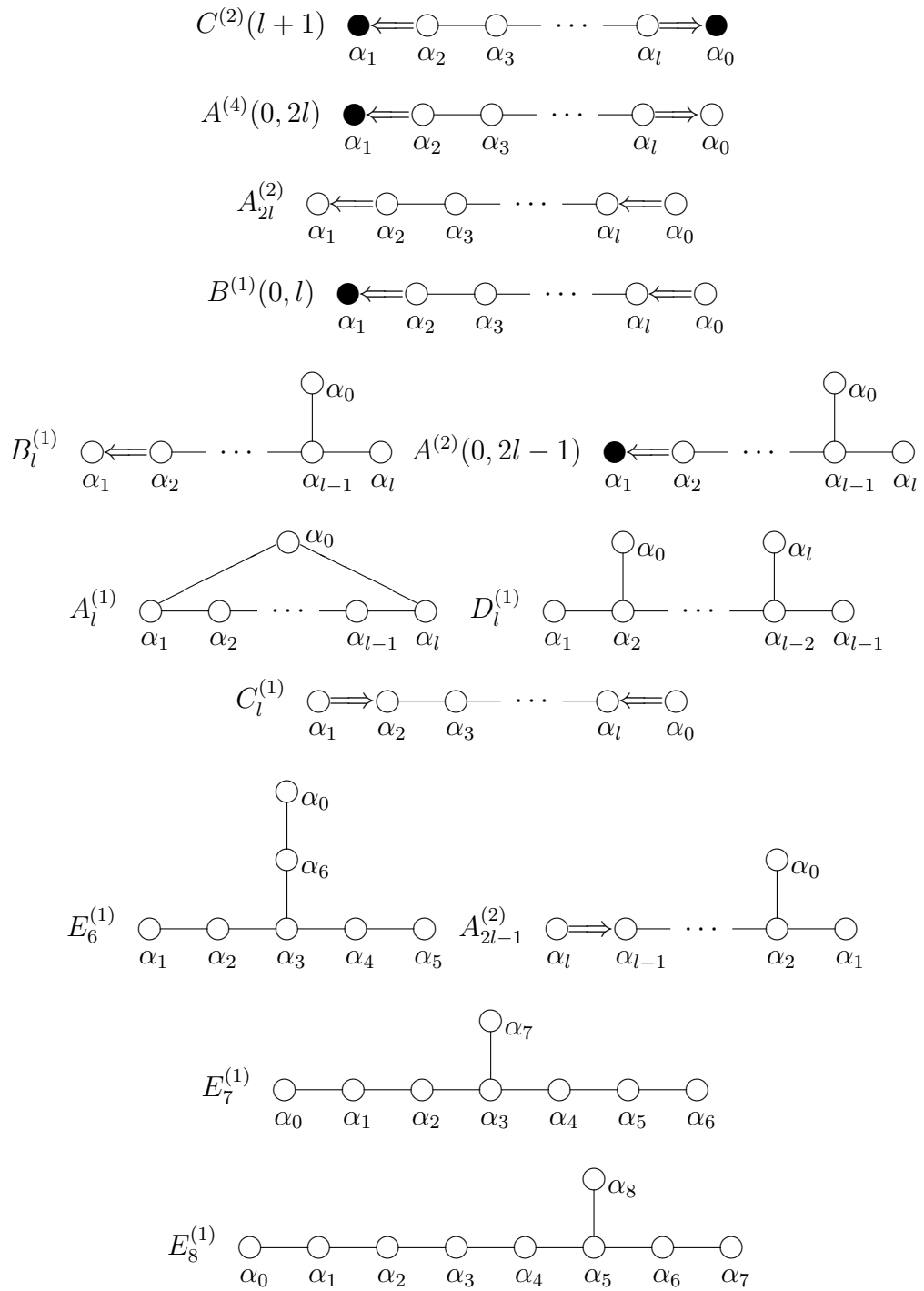
$$\begin{array}{ccc}
 A_1^{(1)} & \begin{array}{c} \alpha_1 \quad \alpha_0 \\ \circ \rightleftarrows \circ \end{array} & A_2^{(2)} & \begin{array}{c} \alpha_1 \quad \alpha_0 \\ \circ \leftarrow \equiv \circ \end{array} \\
 \\
 B^{(1)}(0, 1) & \begin{array}{c} \alpha_1 \quad \alpha_0 \\ \bullet \leftarrow \equiv \circ \end{array} & C^{(2)}(2) & \begin{array}{c} \alpha_1 \quad \alpha_0 \\ \bullet \rightleftarrows \bullet \end{array} & A^{(4)}(0, 2) & \begin{array}{c} \alpha_1 \quad \alpha_0 \\ \bullet \rightleftarrows \circ \end{array}
 \end{array}$$

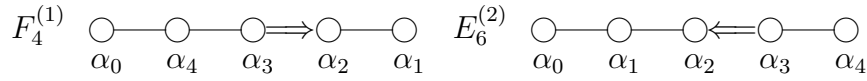
(ii) The case of $l = 2$:

$$\begin{array}{ccc}
 A_2^{(1)} & \begin{array}{c} \alpha_0 \\ \alpha_1 \quad \alpha_2 \\ \circ \quad \circ \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} & C_2^{(1)} & \begin{array}{c} \alpha_2 \quad \alpha_1 \quad \alpha_0 \\ \circ \rightleftarrows \circ \leftarrow \equiv \circ \end{array} & G_2^{(1)} & \begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_0 \\ \circ \leftarrow \equiv \circ \text{---} \circ \end{array} \\
 \\
 A_4^{(2)} & \begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_0 \\ \circ \leftarrow \equiv \circ \leftarrow \equiv \circ \end{array} & D_3^{(2)} & \begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_0 \\ \circ \leftarrow \equiv \circ \rightleftarrows \circ \end{array} & D_4^{(3)} & \begin{array}{c} \alpha_0 \quad \alpha_1 \quad \alpha_2 \\ \circ \text{---} \circ \leftarrow \equiv \circ \end{array} \\
 \\
 B^{(1)}(0, 2) & \begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_0 \\ \bullet \leftarrow \equiv \circ \leftarrow \equiv \circ \end{array} & A^{(2)}(0, 3) & \begin{array}{c} \alpha_2 \quad \alpha_1 \quad \alpha_0 \\ \circ \rightleftarrows \bullet \leftarrow \equiv \circ \end{array} \\
 \\
 C^{(2)}(3) & \begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_0 \\ \bullet \leftarrow \equiv \circ \rightleftarrows \bullet \end{array} & A^{(4)}(0, 4) & \begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_0 \\ \bullet \leftarrow \equiv \circ \rightleftarrows \circ \end{array}
 \end{array}$$

(iii) The case of $l \geq 3$:

$$D_{l+1}^{(2)} \quad \begin{array}{c} \circ \leftarrow \equiv \circ \text{---} \circ \text{---} \dots \text{---} \circ \rightleftarrows \circ \\ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \quad \quad \alpha_l \quad \alpha_0 \end{array}$$





Remark 9.1. (1) K. Saito [13] introduced a notion of elliptic root bases of the reduced-marked elliptic root systems. K. Saito and D. Yoshii [15] gave defining relations of the simply-laced elliptic Lie algebras in terms of the elliptic root bases. The author [18] gave defining relations of $\mathfrak{g}(\Pi, k, g)$ with $(\Pi, k, g) \in T_l^{\text{rm}}$ in terms of the elliptic root base. We can extend the notion to that for $R(\Pi, k, g)$ with $(\Pi, k, g) \in T_l^{\text{nrmm}}$ and we can give defining relations written in terms of that notion. The result will appear elsewhere.

(2) Let $(\Pi, k, g) \in T_l$ with $l \geq 2$. Let $\mathfrak{g}' = [\mathfrak{g}(\Pi, k, g), \mathfrak{g}(\Pi, k, g)]$ and let \mathfrak{z} be the center of \mathfrak{g}' . Using our defining relations, we can show that the canonical map from \mathfrak{g}' to $\mathfrak{g}'/\mathfrak{z}$ is a universal central extension map. Using the map, we can discuss how our algebras relate with the extended affine Lie algebras in [1] and the (2-variable) toroidal Lie algebras in [12]. The result will appear elsewhere.

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